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The quasitriangular structures of biproduct Hopf algebras

ZHAO Li-hui[†1], ZHAO Wen-zheng[2]

([1]Department of Mathematics, Zhejiang University, Hangzhou 310027, China)

([2]Department of Mathematics, Henan Normal University, Xinxiang 453002, China)

[†]E-mail: lihuizhaos@126.com

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Abstract: The construction of the biproduct of Hopf algebras, which consists of smash product and the dual notion of smash coproduct, was first formulated by Radford. In this paper we study the quasitriangular structures over biproduct Hopf algebras $B*H$. We show the necessary and sufficient conditions for biproduct Hopf algebras to be quasitriangular. For the case when they are, we determine completely the unique formula of the quasitriangular structures. And so we find a way to construct solutions of the Yang-Baxter equation over biproduct Hopf algebras in the sense of (Majid, 1990).

Key words: Hopf algebra, Quasitriangular structure, Biproduct
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PRELIMINARIES

The smash product and the smash coproduct are well known in the context of Hopf algebra (Molnar, 1979; Montgomery, 1993) and remain an active area of study (Zhao *et al*., 2000; Delvaux, 2004; Zhao and Wang, 2005). Let H be a Hopf algebra over a field k and suppose B is a left H-module algebra and is also a left H-comodule coalgebra. Radford (1985) found necessary and sufficient conditions for the smash product and the smash coproduct on $B\otimes H$ to make it a Hopf algebra, which is called a biproduct Hopf algebra and denoted by $B*H$.

Majid (1994) found a sufficient condition for a biproduct to become quasitriangular. In this paper, we discuss quasitriangular structures of $B*H$. In Section 2, we show that the formulas of the quasitriangular structures R over $B*H$ if it is a quasitriangular Hopf algebra. In Section 3, we find the necessary and sufficient conditions such that it becomes a quasitriangular Hopf algebra.

Throughout this paper, k denotes an arbitrary field, and $(H, m, \Delta, \varepsilon, S)$ is a Hopf algebra over the field k. We follow the notation in (Montgomery, 1993; Sweedler, 1969), but we will write the comultiplication in H, $\Delta:H\rightarrow H\otimes H$, $\Delta(h)=\sum h_1\otimes h_2$. Denote by ${}_HMod$ the category of left H-modules and by ${}^HMod$ the category of left H-comodules. For (V, ρ) in ${}^HMod$ write: $\rho(v)=\sum v_{-1}\otimes v_0\in H\otimes V$.

We first review some basic facts about algebras and coalgeras in the category ${}^HMod$ and in the category ${}_HMod$.

An algebra A in ${}_HMod$ is a left H-module algebra if the following conditions hold:

(1) $(A, \rightharpoonup)$ is a left H-module structure;

(2) $h\rightharpoonup(ab)=\sum(h_1\rightharpoonup a)(h_2\rightharpoonup b)$ and $h\rightharpoonup 1_A=\varepsilon(h)1_A$.

A coalgebra C in ${}_HMod$ is a left H-module coalgebra if the following conditions hold:

(1) $(C, \rightharpoonup)$ is a left H-module structure;

(2) $\sum(h\rightharpoonup c)_1\otimes(h\rightharpoonup c)_2=\sum h_1\rightharpoonup c_1\otimes h_2\rightharpoonup c_2$ and $\varepsilon(h\rightharpoonup c)=\varepsilon(h)\varepsilon(c)$.

An algebra A in ${}^HMod$ is a left H-comodule algebra if the following conditions hold:

(1) (A, ρ) is a left H-comodule structure;

(2) $\sum(ab)_{-1}\otimes(ab)_0=\sum a_{-1}b_{-1}\otimes a_0b_0$ and $\rho(1_A)=1_H\otimes 1_A$.

A coalgebra C in ${}^HMod$ is a left H-comodule coalgebra if the following conditions hold:

(1) (C, ρ) is a left H-comodule structure;

(2) $\sum (c_1)_{-1}(c_2)_{-1}\otimes(c_1)_0\otimes(c_2)_0=\sum c_{-1}\otimes(c_0)_1\otimes(c_0)_2$ and $\sum c_{-1}\otimes\varepsilon(c_0)=\varepsilon(c)$.

Now we recall the definitions of a biproduct bialgebra and a quasitriangular Hopf algebra.

Proposition 1 (Radford, 1985) Let H be a bialgebra, B be a left H-module algebra and a left H-comodule coalgebra. The tensor product $B\otimes H$ bears a bialgebra structure, if and only if the following conditions hold:

(1) $\Delta_B(1_B)=1_B\otimes 1_B$ and $\varepsilon_B(ab)=\varepsilon_B(a)\varepsilon_B(b)$;

(2) B is a left H-module coalgebra;

(3) B is a left H-comodule algebra;

(i) $\Delta_B(ab)=\sum a_1((a_2)_{-1}\rightharpoonup b_1)\otimes(a_2)_0 b_2$;

(ii) $\sum h_1 b_{-1}\otimes h_2\rightharpoonup b_0=\sum (h_1\rightharpoonup b)_{-1}h_2\otimes(h_1\rightharpoonup b)_0$.

In this case, it is called a biproduct bialgebra and denoted by $B*H$, via the smash product and smash coproduct:

$$m_{B*H}: (B\otimes H)\otimes(B\otimes H)\rightarrow(B\otimes H),$$
$$(a\otimes h)(b\otimes g)=\sum a(h_1\rightharpoonup b)\otimes h_2 g;$$
$$\Delta_{B*H}: B\otimes H\rightarrow(B\otimes H)\otimes(B\otimes H),$$
$$\Delta(b\otimes h)=\sum b_1\otimes(b_2)_{-1}h_1\otimes(b_2)_0\otimes h_2;$$
$$\varepsilon_{B*H}: B\otimes H\rightarrow k,\ \varepsilon(b\otimes h)=\varepsilon_B(b)\varepsilon_H(h).$$

Radford defined such B and H satisfying these conditions as the admissible pair and write it as (B, H).

Proposition 2 (Radford, 1985) Suppose (B, H) is an admissible pair, $B*H$ becomes Hopf algebra if and only if:

(1) H is a Hopf algebra;

(2) The identity I_B has an inverse in the convolution algebra $Hom_k(B, B)$.

Definition 1 (Kassel, 1995; Radford, 1992) A quasitriangular Hopf algebra is a pair (H, R), where H is a Hopf algebra, and $R=\sum R^{(1)}\otimes R^{(2)}\in H\otimes H$, satisfying the following conditions ($r=R$):

$$\text{(QT1) } \sum\varepsilon(R^{(1)})R^{(2)}=\sum R^{(1)}\varepsilon(R^{(2)})=1,$$
$$\text{(QT2) } (\Delta\otimes id)(R)=\sum R^{(1)}\otimes r^{(1)}\otimes R^{(2)}r^{(2)},$$
$$\text{(QT3) } (id\otimes\Delta)(R)=\sum R^{(1)}r^{(1)}\otimes r^{(2)}\otimes R^{(2)},$$
$$\text{(QT4) } \Delta^{\mathrm{cop}}(h)R=R\Delta(h),$$
$$\text{i.e. } \sum R^{(1)}h_1\otimes R^{(2)}h_2=\sum h_2R^{(1)}\otimes h_1R^{(2)}.$$

In this case, R is called a quasitriangular structure over H.

We next introduce three new definitions as follows:

Definition 2 Let H be a Hopf algebra, B be a left H-module algebra with a coalgebra structure. Suppose $V=\sum V^{(1)}\otimes V^{(2)}\in B\otimes H$, $T=\sum T^{(1)}\otimes T^{(2)}\in H\otimes H$, (B, H) is called a compatible pair associated to (T, V) if the following conditions hold ($v=V$):

$$\text{(DC1) } \sum\varepsilon_B(V^{(1)})V^{(2)}=1_H,\ \sum V^{(1)}\varepsilon_H(V^{(2)})=1_B,$$
$$\text{(DC2) } (\Delta_B\otimes id)(V)=\sum V^{(1)}\otimes v^{(1)}\otimes V^{(2)}v^{2)},$$
$$\text{(DC3) } (id\otimes\Delta_H)(V)=\sum V^{(1)}(T^{(1)}\rightharpoonup v^{(1)})\otimes v^{(2)}\otimes V^{(2)}T^{(2)}.$$

If the identity I_B has an inverse S_B in the convolution algebra $Hom_k(B, B)$, then $V^{-1}=\sum S_B(V^{(1)})\otimes V^{(2)}$.

Definition 3 Let H be a Hopf algebra, B be a left H-module algebra with a coalgebra structure. $U=\sum U^{(1)}\otimes U^{(2)}\in H\otimes B$, $T=\sum T^{(1)}\otimes T^{(2)}\in H\otimes H$, (B, H) is called a skew compatible pair associated to (T, U) if the following conditions hold ($u=U$):

$$\text{(SC1) } \sum\varepsilon_H(U^{(1)})U^{(2)}=1_B,\ \sum U^{(1)}\varepsilon_B(U^{(2)})=1_H,$$
$$\text{(SC2) } (\Delta_H\otimes id)(U)=\sum T^{(1)}U^{(1)}\otimes u^{(1)}\otimes U^{(2)}(T^{(2)}\rightharpoonup u^{(2)}),$$
$$\text{(SC3) } (id\otimes\Delta_B)(U)=\sum U^{(1)}u^{(1)}\otimes u^{(2)}\otimes U^{(2)}.$$

Definition 4 Let H be a Hopf algebra, B a left H-module algebra with a coalgebra structure and a left H-comodule structure. Assume that $T=\sum T^{(1)}\otimes T^{(2)}\in H\otimes H$, $V=\sum V^{(1)}\otimes V^{(2)}\in B\otimes H$, $U=\sum U^{(1)}\otimes U^{(2)}\in H\otimes B$ and $Q=\sum Q^{(1)}\otimes Q^{(2)}\in B\otimes B$, (B, Q) is called a weak quasitrianglar pair associated to (T, V, U) if the following conditions hold ($Q=q$, $V=\overline{V}$):

$$\text{(WQT1) } \sum\varepsilon(Q^{(1)})Q^{(2)}=\sum Q^{(1)}\varepsilon(Q^{(2)})=1_B,$$
$$\text{(WQT2) } (\Delta\otimes id)(R)=\sum Q_0^{(1)}V^{(1)}\otimes q^{(1)}\otimes Q^{(2)}(Q_{-1}^{(1)}V^{(2)}\rightharpoonup q^{(2)}),$$
$$\text{(WQT3) } (id\otimes\Delta)(Q)=\sum Q_0{}^{(1)}V^{(1)}(U^{(1)}\rightharpoonup q^{(1)})\otimes q^{(2)}\otimes Q^{(2)}(Q_{-1}^{(1)}V^{(2)}\rightharpoonup U^{(2)}),$$
$$\text{(WQT4) } \sum Q_0^{(1)}V_0^{(1)}\overline{V}^{(1)}(T^{(1)}U^{(1)}\rightharpoonup b_1)\otimes Q^{(2)}(Q_{-2}^{(1)}V^{(2)}\rightharpoonup U^{(2)})(Q_{-1}^{(1)}V_{-1}^{(1)}\overline{V}^{(2)}T^{(2)}\rightharpoonup b_2)=\sum(b_2)_0Q^{(1)}\otimes b_1((b_2)_{-1}\rightharpoonup Q^{(2)}).$$

Example 1 Let (B, Q) be a quasitriangular Hopf algebra, and (H, T) be an arbitrary quasitriangular bialgebra. B is an algebra in ${}_HMod$ with a trivial module action $h \rightharpoonup b=\varepsilon(h)b$. B is also an object in ${}^{H}Mod(\rho(b)=\sum T^{(2)}\otimes T^{(1)}\rightharpoonup b=1_H\otimes b)$. Assume $V=1_B\otimes 1_H$, $U=1_H\otimes 1_B$, then it is easy to see (B, H) is a compatible pair associated to (T, V), (B, H) is a skew compatible pair associated to (T, U), and (B, Q) is a weak quasitriangular pair associated to (T, V, U).

Remark 1 Example 1 shows Definition 4 is a generalization of the usual quasitriangular Hopf algebra.

Example 2 Let (H, T) be a triangular Hopf algebra, and $(B, \bar{Q})$ be a quasitriangular Hopf algebra in ${}_{H}Mod$. Suppose that $Q=\sum T^{(1)}\rightharpoonup \bar{Q}^{(1)}\otimes T^{(2)}\rightharpoonup \bar{Q}^{(2)}$, $U=1\otimes 1$, $V=1\otimes 1$, then (B, Q) is a weak quasitriangular pair associated to (T, V, U).

QUASITRIANGULAR STRUCTURES OVER $B*H$

In this section we will describe the quasitriangular structure over biproduct Hopf algebra $B*H$.

The following lemma is obvious:

Lemma 1 Let $B*H$ be a biproduct Hopf algebra. Define maps as follows:

$$q: B*H\to B, \qquad q(a\otimes h)=\varepsilon(h)a$$
$$p: B*H\to H, \qquad p(a\otimes h)=\varepsilon(a)h$$
$$j: B\to B*H, \qquad j(b)=b\otimes 1_H$$
$$i: H\to B*H, \qquad i(h)= 1_B\otimes h$$

Then

(1) q is a coalgebra map and $q((a\otimes h)(b\otimes g))=a(h\rightharpoonup b)\varepsilon(g)$;

(2) j is an algebra map and $\Delta j(b)=\sum b_1\otimes (b_2)_{-1}\otimes (b_2)_0\otimes 1_H$;

(3) p, i are bialgebra maps.

Let $B*H$ be a biproduct Hopf algebra and $R=\sum R^{(1)}\otimes R^{(2)}\otimes R^{(3)}\otimes R^{(4)}\in B*H\otimes B*H$. Define:

$$T=(p\otimes p)(R)=\sum \varepsilon(R^{(1)})R^{(2)}\otimes \varepsilon(R^{(3)})R^{(4)} =\sum T^{(1)}\otimes T^{(2)}\in H\otimes H,$$

$$Q=(q\otimes q)(R)=\sum R^{(1)}\varepsilon(R^{(2)})\otimes R^{(3)}\varepsilon(R^{(4)}) =\sum Q^{(1)}\otimes Q^{(2)}\in B\otimes B,$$

$$V=(q\otimes p)(R)=\sum R^{(1)}\varepsilon(R^{(2)})\otimes \varepsilon(R^{(3)})R^{(4)} =\sum V^{(1)}\otimes V^{(2)}\in B\otimes H,$$

$$U=(p\otimes q)(R)=\sum \varepsilon(R^{(1)})R^{(2)}\otimes R^{(3)}\varepsilon(R^{(4)}) =\sum U^{(1)}\otimes U^{(2)}\in H\otimes B.$$

Then the following Lemma 2 is obvious:

Lemma 2 With the notions above, let $B*H$ be a biproduct Hopf algebra. If R satisfies condition (QT1), then:

(1) $\sum \varepsilon(T^{(1)})T^{(2)}=\sum T^{(1)}\varepsilon(T^{(2)})=1_H$,

(2) $\sum \varepsilon(Q^{(1)})Q^{(2)}=\sum Q^{(1)}\varepsilon(Q^{(2)})=1_B$,

(3) $\sum \varepsilon(V^{(1)})V^{(2)}=1_H$, $\sum V^{(1)}\varepsilon(V^{(2)})=1_B$,

(4) $\sum \varepsilon(U^{(1)})U^{(2)}=1_B$, $\sum U^{(1)}\varepsilon(U^{(2)})=1_H$.

Lemma 3 Let $B*H$ be a biproduct Hopf algebra and $R=\sum R^{(1)}\otimes R^{(2)}\otimes R^{(3)}\otimes R^{(4)}\in B*H\otimes B*H$. If $(B*H, R)$ is a quasitriangular, then we have:

(1) $\sum \varepsilon(R^{(1)})R^{(2)}\otimes R^{(3)}\otimes R^{(4)}=\sum T^{(1)}U^{(1)}\otimes U^{(2)}\otimes T^{(2)}$,

(2) $\sum R^{(1)}\otimes \varepsilon(R^{(2)})R^{(3)}\otimes R^{(4)} =\sum V^{(1)}(T^{(1)}\rightharpoonup Q^{(1)})\otimes Q^{(2)}\otimes V^{(2)}T^{(2)}$,

(3) $\sum R^{(1)}\otimes R^{(2)}\otimes \varepsilon(R^{(3)})R^{(4)}=\sum V^{(1)}\otimes T^{(1)}\otimes V^{(2)}T^{(2)}$,

(4) $\sum R^{(1)}\otimes R^{(2)}\otimes R^{(3)}\varepsilon(R^{(4)}) =\sum V^{(1)}(T^{(1)}\rightharpoonup Q^{(1)})\otimes U^{(1)}\otimes Q^{(2)}(V^{(2)}T^{(2)}\rightharpoonup U^{(2)})$,

(5) $R=\sum \bar{V}^{(1)}(T^{(1)}\rightharpoonup V^{(1)}(t^{(1)}\rightharpoonup Q^{(1)}))\otimes \bar{T}^{(1)}U^{(1)} \otimes Q^{(2)}(V^{(2)}t^{(2)}\rightharpoonup U^{(2)})\otimes \bar{V}^{(2)}T^{(2)}\bar{T}^{(2)}$,

where $\bar{V}=V$, $t=\bar{T}=T$.

Proof By (QT2) and (QT3), we have:

$$\begin{aligned}&\sum R_1^{(1)}\otimes (R_2^{(1)})_{-1}R_1^{(2)}\otimes (R_2^{(1)})_0\otimes R_2^{(2)}\otimes R_1^{(3)} \otimes (R_2^{(3)})_{-1}R_1^{(4)}\otimes (R_2^{(3)})_0\otimes R_2^{(4)}\\ &=\sum R^{(1)}(R_1^{(2)}\rightharpoonup \bar{r}^{(1)})\otimes R_2^{(2)}\bar{r}^{(2)}\otimes \bar{R}^{(1)}(\bar{R}_1^{(2)}\rightharpoonup r^{(1)}) \otimes \bar{R}_2^{(2)}r^{(2)}\otimes \bar{r}^{(3)}(\bar{r}_1^{(4)}\rightharpoonup r^{(3)})\otimes \bar{r}_2^{(4)}r^{(4)}\\ &\quad\otimes R^{(3)}(R_1^{(4)}\rightharpoonup \bar{R}^{(3)})\otimes R_2^{(4)}\bar{R}^{(4)},\end{aligned} \tag{1}$$

where $r=\bar{r}=R=\bar{R}$.

By applying $p\otimes p\otimes q\otimes p$, $q\otimes p\otimes p\otimes p$, $q\otimes q\otimes q\otimes p$ and $q\otimes p\otimes q\otimes q$ to both sides of Eq.(1) respectively, we can obtain (1), (3), (2) and (4); applying

$q\otimes p\otimes q\otimes p$ to both sides of Eq.(1) and by (2), (3), we can obtain (5).

Proposition 3 Let $B*H$ be a biproduct Hopf algebra and $R=\sum R^{(1)}\otimes R^{(2)}\otimes R^{(3)}\otimes R^{(4)}\in B*H\otimes B*H$. If $(B*H, R)$ is a quasitriangular Hopf algebra, then we have:

$$(1)\ \sum R^{(1)}(R^{(2)}\rightharpoonup b)\otimes\varepsilon(R^{(3)})R^{(4)} = \sum b_0R^{(1)}\otimes\varepsilon(R^{(2)})\varepsilon(R^{(3)})b_{-1}R^{(4)},$$

$$(2)\ R=\sum Q_0^{(1)}V_0^{(1)}\overline{V}^{(1)}\otimes T^{(1)}U^{(1)}\otimes Q^{(2)}(Q_{-2}^{(1)}V^{(2)}\rightharpoonup U^{(2)})\otimes Q_{-1}^{(1)}V_{-1}^{(1)}\overline{V}^{(2)}T^{(2)},$$

where $b\in B$, $v=\overline{V}$.

Proof Firstly, let h=1 in (QT4), then we obtain (1). Secondly, applying $q\otimes p\otimes q\otimes p$ to both sides of Eq.(1) and using (1), we can obtain (2).

BIPRODUCT BECOMES QUASITRIANGULAR

Proposition 4 Let $B*H$ be a biproduct Hopf algebra. Assume that $R=\sum Q_0^{(1)}V_0^{(1)}\overline{V}^{(1)}\otimes T^{(1)}U^{(1)}\otimes Q^{(2)}(Q_{-2}^{(1)}V^{(2)}\rightharpoonup U^{(2)})\otimes Q_{-1}^{(1)}V_{-1}^{(1)}\overline{V}^{(2)}T^{(2)}$ is a quasitriangular structure over $B*H$, where $T=\sum T^{(1)}\otimes T^{(2)}\in H\otimes H$, $Q=\sum Q^{(1)}\otimes Q^{(2)}\in B\otimes B$, $V=\sum V^{(1)}\otimes V^{(2)}\in B\otimes H$, $U=\sum U^{(1)}\otimes U^{(2)}\in H\otimes B$. Then we have the following identities ($v=V$):

$$(\mathrm{C1})\sum V^{(1)}(T^{(1)}\rightharpoonup b)\otimes V^{(2)}T^{(2)}=\sum b_0V^{(1)}\otimes b_{-1}V^{(2)},$$

$$(\mathrm{C2})\sum T^{(1)}U^{(1)}b_{-1}\otimes U^{(2)}(T^{(2)}\rightharpoonup b_0)=\sum U^{(1)}\otimes bU^{(2)},$$

$$(\mathrm{C3})\sum V^{(1)}\otimes V^{(2)}h=\sum h_2\rightharpoonup V^{(1)}\otimes h_1V^{(2)},$$

$$(\mathrm{C4})\sum U^{(1)}h\otimes U^{(2)}=\sum h_2U^{(1)}\otimes h_1\rightharpoonup U^{(2)},$$

$$(\mathrm{C5})\sum Q^{(1)}\otimes Q^{(2)}\varepsilon(h)=\sum h_2\rightharpoonup Q^{(1)}\otimes h_1\rightharpoonup Q^{(2)},$$

$$(\mathrm{C6})\sum V_{-1}^{(1)}T^{(1)}\otimes V_0^{(1)}\otimes V^{(2)}T^{(2)}=\sum T^{(1)}\otimes V^{(1)}\otimes T^{(2)}V^{(2)},$$

$$(\mathrm{C7})\sum T^{(1)}U^{(1)}\otimes U_{-1}^{(2)}T^{(2)}\otimes U_0^{(2)}=\sum U^{(1)}T^{(1)}\otimes T^{(2)}\otimes U^{(2)},$$

$$(\mathrm{C8})\sum Q_0^{(1)}V^{(1)}\otimes Q_{-1}^{(2)}Q_{-1}^{(1)}V^{(2)}\otimes Q_0^{(2)}=\sum Q_0^{(1)}V^{(1)}(U^{(1)}\rightharpoonup v^{(1)})\otimes v^{(2)}\otimes Q^{(2)}(Q_{-1}^{(1)}V^{(2)}\rightharpoonup U^{(2)}),$$

$$(\mathrm{C9})\sum (Q_0^{(1)}V^{(1)})_{-1}U^{(1)}\otimes(Q_0^{(1)}V^{(1)})_0\otimes Q^{(2)}(Q_{-1}^{(1)}V^{(2)}\rightharpoonup U^{(2)})=\sum T^{(1)}U^{(1)}\otimes Q^{(1)}\otimes U^{(2)}(T^{(2)}\rightharpoonup Q^{(2)}).$$

Proof (1) By (QT4), for any $b\in B$, $h\in H$ we have:

$$\begin{aligned}&\sum Q_0^{(1)}V_0^{(1)}\overline{V}^{(1)}((T^{(1)}U^{(1)})_1\rightharpoonup b_1)\otimes (T^{(1)}U^{(1)})_2(b_2)_{-1}h_1\otimes Q^{(2)}(Q_{-2}^{(1)}V^{(2)}\rightharpoonup U^{(2)})\\&((Q_{-1}^{(1)}V_{-1}^{(1)}\overline{V}^{(2)}T^{(2)})_1\rightharpoonup(b_2)_0)\otimes(Q_{-1}^{(1)}V_{-1}^{(1)}\overline{V}^{(2)}T^{(2)})_2h_2\\&=\sum(b_2)_0(h_2\rightharpoonup Q_0^{(1)}V_0^{(1)}\overline{V}^{(1)})\otimes h_3T^{(1)}U^{(1)}\otimes b_1(((b_2)_{-1}h_1)_1\rightharpoonup Q^{(2)}(Q_{-2}^{(1)}V^{(2)}\rightharpoonup U^{(2)}))\\&\otimes((b_2)_{-1}h_1)_2Q_{-1}^{(1)}V_{-1}^{(1)}\overline{V}^{(2)}T^{(2)}.\end{aligned}\tag{2}$$

Let h=1 in Eq.(2), we obtain (C1); By applying $p\otimes q$ to both sides of Eq.(2), we obtain (C2). Let b=1 in Eq.(2) and using p, q, we obtain (C3), (C4) and (C5).

(2) By (QT2), we have

$$\begin{aligned}&\sum(Q_0^{(1)}V_0^{(1)}\overline{V}^{(1)})_1\otimes((Q_0^{(1)}V_0^{(1)}\overline{V}^{(1)})_2)_{-1}(T^{(1)}U^{(1)})_1\\&\otimes((Q_0^{(1)}V_0^{(1)}\overline{V}^{(1)})_2)_0\otimes(T^{(1)}U^{(1)})_2\otimes\\&Q^{(2)}(Q_{-2}^{(1)}V^2\rightharpoonup U^{(2)})\otimes Q_{-1}^{(1)}V_{-1}^{(1)}\overline{V}^{(2)}T^{(2)}\\&=Q_0^{(1)}V_0^{(1)}\overline{V}^{(1)}\otimes T^{(1)}U^{(1)}\otimes q_0^{(1)}v_0^{(1)}\overline{v}^{(1)}\otimes\\&t^{(1)}u^{(1)}\otimes Q^{(2)}(Q_{-2}^{(1)}V^{(2)}\rightharpoonup U^{(2)})\\&((Q_{-1}^{(1)}V_{-1}^{(1)}\overline{V}^{(2)}T^{(2)})_1\rightharpoonup q^{(2)}(q_{-2}^{(1)}v^{(2)}\rightharpoonup u^{(2)}))\otimes\\&(Q_{-1}^{(1)}V_{-1}^{(1)}\overline{V}^{(2)}T^{(2)})_2q_{-1}^{(1)}v_{-1}^{(1)}\overline{v}^{(2)}t^{(2)}.\end{aligned}\tag{3}$$

By applying $p\otimes q\otimes q$ and $p\otimes q\otimes p$ to both sides of Eq.(3), we can obtain (C9) and (C6).

(3) By (QT3), we have:

$$\begin{aligned}&\sum Q_0^{(1)}V_0^{(1)}\overline{V}^{(1)}\otimes T^{(1)}U^{(1)}\otimes(Q^{(2)}(Q_{-2}^{(1)}V^{(2)}\rightharpoonup U^{(2)}))_1\\&\otimes((Q^{(2)}\ (Q_{-2}^{(1)}V^{(2)}\rightharpoonup U^{(2)}))_2)_{-1}(Q_{-1}^{(1)}V_{-1}^{(1)}\overline{V}^{(2)}T^{(2)})_1\\&\otimes((Q^{(2)}(Q_{-2}^{(1)}V^{(2)}\rightharpoonup U^{(2)}))_2)_0\otimes(Q_{-1}^{(1)}V_{-1}^{(1)}\overline{V}^{(2)}T^{(2)})_2\\&=\sum(Q_0^{(1)}V_0^{(1)}\overline{V}^{(1)})((T^{(1)}U^{(1)})_1\rightharpoonup q_0^{(1)}v_0^{(1)}\overline{v}^{(1)})\\&\otimes(T^{(1)}U^{(1)})_2t^{(1)}u^{(1)}\otimes q^{(2)}(q_{-2}^{(1)}v^{(2)}\rightharpoonup u^{(2)})\\&\otimes q_{-1}^{(1)}v_{-1}^{(1)}\overline{v}^{(2)}t^{(2)}\otimes Q^{(2)}(Q_{-2}^{(1)}V^{(2)}\rightharpoonup U^{(2)})\\&\otimes Q_{-1}^{(1)}V_{-1}^{(1)}\overline{V}^{(2)}T^{(2)},\end{aligned}\tag{4}$$

where $\overline{V}=V=v=\overline{v}$, $q=Q$, $T=t$ and $u=U$.

By applying $q\otimes p\otimes q$ and $p\otimes p\otimes q$ to both sides of Eq.(4), we can obtain (C8) and (C7). The proof is complete.

Remark 2 If $B*H$ is a quasitriangular Hopf algebra, applying (C1), we see that the formulas of quasitriangular structures R in Lemma 3 and Proposition 3 are equivalent.

Remark 3 The condition (C1) says if $B*H$ is a quasitriangular Hopf algebra, then the comodule structure over B depends on its module structure.

Remark 4 If the condition (C1) holds then (DC3), (WQT2), (WQT3) and (WQT4) can be also expressed as follows:

$$(\text{DC3})' \sum V^{(1)}\otimes V_1^{(2)}\otimes V_2^{(2)}=\sum v_0^{(1)}V^{(1)}\otimes v^{(2)}\otimes v_{-1}^{(1)}V^{(2)},$$

$$(\text{WQT2})' \sum Q_1^{(1)}\otimes Q_2^{(1)}\otimes Q^{(2)}=\sum V^{(1)}(T^{(1)}\rightharpoonup Q^{(1)})\otimes q^{(1)}\otimes Q^{(2)}(V^{(2)}T^{(2)}\rightharpoonup q^{(2)}),$$

$$(\text{WQT3})' \sum Q^{(1)}\otimes Q_1^{(2)}\otimes Q_2^{(2)}=\sum V^{(1)}(T^{(1)}\rightharpoonup Q^{(1)})(U^{(1)}\rightharpoonup q^{(1)})\otimes Q^{(1)}(V^{(2)}T^{(2)}\rightharpoonup U^{(2)}),$$

$$(\text{WQT4})' \sum V^{(1)}(T^{(1)}\rightharpoonup v^{(1)}(t^{(1)}\rightharpoonup Q^{(1)})(U^{(1)}\rightharpoonup b_1))\otimes Q^{(2)}(v^{(2)}t^{(2)}\rightharpoonup U^{(2)})(V^{(2)}T^{(2)}\rightharpoonup b_2)=\sum (b_2)_0Q^{(1)}\otimes b_1((b_2)_{-1}\rightharpoonup Q^{(2)}).$$

Proposition 5 Let $B*H$ be a biproduct Hopf algebra. Assume that $R=\sum Q_0^{(1)}V_0^{(1)}\overline{V}^{(1)}\otimes T^{(1)}U^{(1)}\otimes Q^{(2)}(Q_{-2}^{(1)}V^{(2)}\rightharpoonup U^{(2)})\otimes Q_{-1}^{(1)}V_{-1}^{(1)}\overline{V}^{(2)}T^{(2)}$ is a quasitriangular structure over $B*H$, where $T=\sum T^{(1)}\otimes T^{(2)}\in H\otimes H$, $Q=\sum Q^{(1)}\otimes Q^{(2)}\in B\otimes B$, $V=\sum V^{(1)}\otimes V^{(2)}\in B\otimes H$, $U=\sum U^{(1)}\otimes U^{(2)}\in H\otimes B$, then we have:

(1) (H, T) is a quasitriangular Hopf algebra;

(2) (B, H) is a compatible pair associated to (T, V);

(3) (B, H) is a skew compatible pair associated to (T, U);

(4) (B, Q) is a weak quasitriangular pair associated to (T, V, U).

Proof (1) By Lemma 1, p is a bialgebra map. Thus, $T=(p\otimes p)(R)=\sum T^{(1)}\otimes T^{(2)}$ is a quasitriangular structure over H.

(2) Applying $p\otimes p\otimes q$ to both sides of Eq.(3), we can get (DC2); applying $q\otimes p\otimes p$ to both sides of Eq.(4), we can get (DC3). Thus, (B, H) is a compatible pair associated to (T, V) by Lemma 2.

(3) Applying $p\otimes p\otimes q$ to both sides of Eq.(3), we can get (SC2); applying $p\otimes q\otimes q$ to both sides of Eq.(4), we can get (SC3). Thus, (B, H) is a skew compatible pair associated to (T, U) by Lemma 2.

(4) Applying $q\otimes q\otimes q$ to both sides of Eq.(3), we can get (WQT2); applying $q\otimes q\otimes q$ to both sides of Eq.(4), we can get (WQT3); applying $q\otimes q$ to both sides of Eq.(2), we can obtain (WQT4). Thus (B, Q) is a weak quasitriangular pair associated to (T, V, U) by Lemma 2.

These complete the proof.

Proposition 6 Let $B*H$ be a biproduct Hopf algebra. If there exist elements $T=\sum T^{(1)}\otimes T^{(2)}\in H\otimes H$, $Q=\sum Q^{(1)}\otimes Q^{(2)}\in B\otimes B$, $V=\sum V^{(1)}\otimes V^{(2)}\in B\otimes H$, $U=\sum U^{(1)}\otimes U^{(2)}\in H\otimes B$, such that the following conditions hold:

(1) (H, T) is a quasitriangular Hopf algebra;

(2) (B, H) is a compatible pair associated to (T, V);

(3) (B, H) is a skew compatible pair associated to (T, U);

(4) (B, Q) is a weak quasitriangular pair associated to (V, U);

(5) T, V, U, Q satisfy the conditions (C1)~(C9) in Proposition 4.

Then $(B*H, R)$ is a quasitriangular Hopf algebra with a quasitriangular structure given by $(V=\overline{V})$:

$$R=\sum Q_0^{(1)}V_0^{(1)}\overline{V}^{(1)}\otimes T^{(1)}U^{(1)}\otimes Q^{(2)}(Q_{-2}^{(1)}V^{(2)}\rightharpoonup U^{(2)})\otimes Q_{-1}^{(1)}V_{-1}^{(1)}\overline{V}^{(2)}T^{(2)}.$$

Proof By the definition of R, denote $R=\sum R^{(1)}\otimes R^{(2)}$, where $R^{(1)}=\sum Q_0^{(1)}V_0^{(1)}\overline{V}^{(1)}\otimes T^{(1)}U^{(1)}$, $R^{(2)}=\sum Q^{(2)}(Q_{-2}^{(1)}V^{(2)}\rightharpoonup U^{(2)})\otimes Q_{-1}^{(1)}V_{-1}^{(1)}\overline{V}^{(2)}T^{(2)}$, then by Lemma 2, we have:

$$\sum\varepsilon(R^{(1)})R^{(2)}=\sum\varepsilon(Q_0^{(1)}V_0^{(1)}\overline{V}^{(1)})\varepsilon(T^{(1)}U^{(1)})Q^{(2)}(Q_{-2}^{(1)}V^{(2)}\rightharpoonup U^{(2)})\otimes Q_{-1}^{(1)}V_{-1}^{(1)}\overline{V}^{(2)}T^{(2)}=1_B\otimes 1_H,$$

$$\sum R^{(1)}\varepsilon(R^{(2)})=\sum Q_0^{(1)}V_0^{(1)}\overline{V}^{(1)}\otimes T^{(1)}U^{(1)}\varepsilon(Q^{(2)}(Q_{-2}^{(1)}V^{(2)}\rightharpoonup U^{(2)}))\varepsilon(Q_{-1}^{(1)}V_{-1}^{(1)}\overline{V}^{(2)}T^{(2)})=1_B\otimes 1_H.$$

So (QT1) is checked.

In what follows, we first check that (QT2) holds:

$(\Delta\otimes id)(R)$

$=\sum (Q_0^{(1)}V_0^{(1)}\overline{V}^{(1)})_1\otimes((Q_0^{(1)}V_0^{(1)}\overline{V}^{(1)})_2)_{-1}$

$(T^{(1)}U^{(1)})_1\otimes((Q_0^{(1)}V_0^{(1)}\overline{V}^{(1)})_2)_0\otimes(T^{(1)}U^{(1)})_2\otimes Q^{(2)}$

$(Q_{-2}^{(1)}V^{(2)}\rightharpoonup U^{(2)})\otimes Q_{-1}^{(1)}V_{-1}^{(1)}\overline{V}^{(2)}T^{(2)}$

$=\sum (Q_0^{(1)}V_0^{(1)})_1(((Q_0^{(1)}V_0^{(1)})_2)_{-1}\rightharpoonup\overline{V}_1^{(1)})$

$\otimes(((Q_0^{(1)}V_0^{(1)})_2)_0\overline{V}_2^{(1)})_{-1}T^{(1)}\overline{T}^{(1)}U^{(1)}$

$\otimes(((Q_0^{(1)}V_0^{(1)})_2)_0\overline{V}_2^{(1)})_0\otimes t^{(1)}u^{(1)}\otimes Q^{(2)}(Q_{-2}^{(1)}V^{(2)}\rightharpoonup$

$U^{(2)}(\overline{T}^{(2)}\rightharpoonup u^{(2)}))\otimes Q_{-1}^{(1)}V_{-1}^{(1)}\overline{V}^{(2)}T^{(2)}t^{(2)}$

by (QT2), (SC2), (i)

$=\sum (Q_0^{(1)}V_0^{(1)})_1(((Q_0^{(1)}V_0^{(1)})_2)_{-2}\rightharpoonup\overline{V}^{(1)})$

$\otimes((Q_0^{(1)}V_0^{(1)})_2)_{-1}T^{(1)}\overline{T}^{(1)}U^{(1)}\otimes((Q_0^{(1)}V_0^{(1)})_2)_0v^{(1)}$

$\otimes t^{(1)}u^{(1)}\otimes Q^{(2)}(Q_{-2}^{(1)}V^{(2)}\rightharpoonup U^{(2)}(\overline{T}^{(2)}\rightharpoonup u^{(2)}))$

$\otimes Q_{-1}^{(1)}V_{-1}^{(1)}\overline{V}^{(2)}T^{(2)}v^{(2)}t^{(2)}$ by (DC2), (C6)

$=\sum (Q_0^{(1)})_1(((Q_0^{(1)})_2)_{-1}\rightharpoonup V_0^{(1)})((((Q_0^{(1)})_2)_0\bar{v}_0^{(1)})_{-2}\rightharpoonup$

$\overline{V}^{(1)})\otimes(((Q_0^{(1)})_2)_0\bar{v}_0^{(1)})_{-1}T^{(1)}\overline{T}^{(1)}U^{(1)}$

$\otimes(((Q_0^{(1)})_2)_0\bar{v}_0^{(1)})_0v^{(1)}\otimes t^{(1)}u^{(1)}\otimes Q^{(2)}(Q_{-2}^{(1)}V^{(2)}\bar{v}^{(2)}$

$\rightharpoonup U^{(2)}(\overline{T}^{(2)}\rightharpoonup u^{(2)}))\otimes Q_{-1}^{(1)}V_{-1}^{(1)}\bar{v}_{-1}^{(1)}\overline{V}^{(2)}T^{(2)}v^{(2)}t^{(2)}$

by (i), (DC2)

$=\sum (Q_1^{(1)})_0((Q_2^{(1)})_{-3}\rightharpoonup V_0^{(1)})((Q_2^{(1)})_{-2}\bar{v}_{-2}^{(1)}\rightharpoonup\overline{V}^{(1)})$

$\otimes(Q_2^{(1)})_{-1}\bar{v}_{-1}^{(1)}T^{(1)}\overline{T}^{(1)}U^{(1)}\otimes(Q_2^{(1)})_0\bar{v}_0^{(1)}v^{(1)}\otimes t^{(1)}u^{(1)}$

$\otimes Q^{(2)}((Q_1^{(1)})_{-2}(Q_2^{(1)})_{-5}V^{(2)}\bar{v}^{(2)}\rightharpoonup U^{(2)}(\overline{T}^{(2)}\rightharpoonup$

$u^{(2)}))\otimes(Q_1^{(1)})_{-1}(Q_2^{(1)})_{-4}V_{-1}^{(1)}\bar{v}_{-3}^{(1)}\overline{V}^{(2)}T^{(2)}v^{(2)}t^{(2)}$

by comodule coalgebra

$=\sum (Q_1^{(1)})_0((Q_2^{(1)})_{-4}\rightharpoonup V^{(1)})_0((Q_2^{(1)})_{-2}\bar{v}_{-2}^{(1)}\rightharpoonup\overline{V}^{(1)})$

$\otimes(Q_2^{(1)})_{-1}\bar{v}_{-1}^{(1)}T^{(1)}\overline{T}^{(1)}U^{(1)}\otimes(Q_2^{(1)})_0\bar{v}_0^{(1)}v^{(1)}\otimes t^{(1)}u^{(1)}$

$\otimes Q^{(2)}((Q_1^{(1)})_{-2}(Q_2^{(1)})_{-5}V^{(2)}\bar{v}^{(2)}\rightharpoonup U^{(2)}(\overline{T}^{(2)}\rightharpoonup u^{(2)}))$

$\otimes(Q_1^{(1)})_{-1}((Q_2^{(1)})_{-4}\rightharpoonup V^{(1)})_{-1}(Q_2^{(1)})_{-3}\bar{v}_{-3}^{(1)}\overline{V}^{(2)}T^{(2)}v^{(2)}t^{(2)}$

by (ii)

$=\sum (Q_1^{(1)})_0V_0^{(1)}\overline{V}^{(1)}\otimes T^{(1)}((Q_2^{(1)})_{-1}\bar{v}_{-1}^{(1)})_1\overline{T}^{(1)}U^{(1)}$

$\otimes(Q_2^{(1)})_0\bar{v}_0^{(1)}v^{(1)}\otimes t^{(1)}u^{(1)}$

$\otimes Q^{(2)}((Q_1^{(1)})_{-2}V^{(2)}(Q_2^{(1)})_{-2}\bar{v}^{(2)}\rightharpoonup U^{(2)}(\overline{T}^{(2)}\rightharpoonup$

$u^{(2)}))\otimes(Q_1^{(1)})_{-1}V_{-1}^{(1)}\overline{V}^{(2)}T^{(2)}((Q_2^{(1)})_{-1}\bar{v}_{-1}^{(1)})_2v^{(2)}t^{(2)}$

by (C3) (QT4)

$=\sum (Q_1^{(1)})_0V_0^{(1)}\overline{V}^{(1)}\otimes T^{(1)}(Q_2^{(1)})_{-2}\bar{v}_{-2}^{(1)}\overline{T}^{(1)}U^{(1)}$

$\otimes(Q_2^{(1)})_0\bar{v}_0^{(1)}v^{(1)}\otimes t^{(1)}u^{(1)}$

$\otimes Q^{(2)}(((Q_1^{(1)})_{-2}V^{(2)}(Q_2^{(1)})_{-3}\bar{v}^{(2)})_1\rightharpoonup U^{(2)})$

$(((Q_1^{(1)})_{-2}V^{(2)}(Q_2^{(1)})_{-3}\bar{v}^{(2)})_2\overline{T}^{(2)}\rightharpoonup u^{(2)})$

$\otimes(Q_1^{(1)})_{-1}V_{-1}^{(1)}\overline{V}^{(2)}T^{(2)}(Q_2^{(1)})_{-1}\bar{v}_{-1}^{(1)}v^{(2)}t^{(2)}$

by module algebra

$=\sum (Q_1^{(1)})_0V_0^{(1)}\overline{V}^{(1)}\otimes T^{(1)}(Q_2^{(1)})_{-2}(\bar{v}_0^{(1)}\widetilde{V}^{(1)})_{-2}\overline{T}^{(1)}U^{(1)}$

$\otimes(Q_2^{(1)})_0(\bar{v}_0^{(1)}\widetilde{V}^{(1)})_0v^{(1)}\otimes t^{(1)}u^{(1)}$

$\otimes Q^{(2)}(((Q_1^{(1)})_{-2})_1V_1^{(2)}((Q_2^{(1)})_{-3})_1\bar{v}^{(2)}\rightharpoonup$

$U^{(2)})(((Q_1^{(1)})_{-2})_2V_2^{(2)}((Q_2^{(1)})_{-3})_2\bar{v}_{-1}^{(1)}\widetilde{V}^{(2)}\overline{T}^{(2)}\rightharpoonup u^{(2)})$

$\otimes(Q_1^{(1)})_{-1}V_{-1}^{(1)}\overline{V}^{(2)}T^{(2)}(Q_2^{(1)})_{-1}(\bar{v}_0^{(1)}\widetilde{V}^{(1)})_{-1}v^{(2)}t^{(2)}$

by (DC3)′

$=\sum (Q_1^{(1)})_0V_0^{(1)}\overline{V}^{(1)}\otimes T^{(1)}(Q_2^{(1)})_{-2}\bar{v}_{-2}^{(1)}\widetilde{V}_{-2}^{(1)}\overline{T}^{(1)}U^{(1)}$

$\otimes(Q_2^{(1)})_0\bar{v}_0^{(1)}\widetilde{V}_0^{(1)}v^{(1)}\otimes t^{(1)}u^{(1)}$

$\otimes Q^{(2)}(((Q_1^{(1)})_{-2})_1V_1^{(2)}((Q_2^{(1)})_{-3})_1\bar{v}^{(2)}\rightharpoonup U^{(2)})$

$(((Q_1^{(1)})_{-2})_2V_2^{(2)}((Q_2^{(1)})_{-3})_2\bar{v}_{-3}^{(1)}\widetilde{V}^{(2)}\overline{T}^{(2)}\rightharpoonup u^{(2)})$

$\otimes(Q_1^{(1)})_{-1}V_{-1}^{(1)}\overline{V}^{(2)}T^{(2)}(Q_2^{(1)})_{-1}\bar{v}_{-1}^{(1)}\widetilde{V}_{-1}^{(1)}v^{(2)}t^{(2)}$

by comodule algebra

$=\sum (Q_1^{(1)})_0V_0^{(1)}\overline{V}^{(1)}\otimes T^{(1)}\overline{T}^{(1)}(Q_2^{(1)})_{-3}\bar{v}_{-3}^{(1)}U^{(1)}$

$\otimes(Q_2^{(1)})_0\bar{v}_0^{(1)}\widetilde{V}_0^{(1)}v^{(1)}\otimes t^{(1)}u^{(1)}$

$\otimes Q^{(2)}(((Q_1^{(1)})_{-2})_1V_1^{(2)}(Q_2^{(1)})_{-4}\bar{v}^{(2)}\rightharpoonup U^{(2)})$

$(((Q_1^{(1)})_{-2})_2V_2^{(2)}\overline{T}^{(2)}(Q_2^{(1)})_{-2}\bar{v}_{-2}^{(1)}\widetilde{V}^{(2)}\rightharpoonup u^{(2)})$

$\otimes(Q_1^{(1)})_{-1}V_{-1}^{(1)}\overline{V}^{(2)}T^{(2)}(Q_2^{(1)})_{-1}\bar{v}_{-1}^{(1)}\widetilde{V}_{-1}^{(1)}v^{(2)}t^{(2)}$

by (C6), (QT4)

$=\sum Q_0^{(1)}V_0^{(1)}\overline{V}^{(1)}\otimes T^{(1)}\overline{T}^{(1)}q_{-3}^{(1)}\bar{v}_{-3}^{(1)}U^{(1)}\otimes q_0^{(1)}\bar{v}_0^{(1)}\widehat{V}_0^{(1)}v^{(1)}$

$\otimes t^{(1)}u^{(1)}\otimes Q^{(2)}(Q_{-3}^{(1)}V_1^{(2)}\rightharpoonup q^{(2)})$

$((Q_{-2}^{(1)})_1V_2^{(2)}q_{-4}^{(1)}\bar{v}^{(2)}\rightharpoonup U^{(2)})$

$((Q_{-2}^{(1)})_2V_3^{(2)}\overline{T}^{(2)}q_{-2}^{(1)}\bar{v}_{-2}^{(1)}\widehat{V}^{(2)}\rightharpoonup u^{(2)})$

$\otimes Q_{-1}^{(1)}V_{-1}^{(1)}\overline{V}^{(2)}T^{(2)}q_{-1}^{(1)}\bar{v}_{-1}^{(1)}\widehat{V}_{-1}^{(1)}v^{(2)}t^{(2)}$

by (WQT2), (DC3)′

$=\sum Q_0^{(1)}V_0^{(1)}\overline{V}^{(1)}\otimes T^{(1)}\overline{T}^{(1)}q_{-3}^{(1)}\bar{v}_{-3}^{(1)}U^{(1)}\otimes q_0^{(1)}\bar{v}_0^{(1)}\widehat{V}_0^{(1)}v^{(1)}$

$\otimes t^{(1)}u^{(1)}\otimes Q^{(2)}(Q_{-3}^{(1)}V_1^{(2)}\rightharpoonup q^{(2)}(q_{-4}^{(1)}\bar{v}^{(2)}\rightharpoonup U^{(2)}))$

$$(Q_{-2}^{(1)}V_2^{(2)}\overline{T}^{(2)}q_{-2}^{(1)}\overline{v}_{-2}^{(1)}\widehat{V}^{(2)}\rightharpoonup u^{(2)})$$

$$\otimes Q_{-1}^{(1)}V_{-1}^{(1)}\overline{V}^{(2)}T^{(2)}q_{-1}^{(1)}\overline{v}_{-1}^{(1)}\widehat{V}_{-1}^{(1)}v^{(2)}t^{(2)}$$

by module algebra

$$=\sum Q_0^{(1)}V_0^{(1)}\overline{V}^{(1)}\otimes T^{(1)}\overline{T}^{(1)}U^{(1)}\otimes q_0^{(1)}\widehat{V}_0^{(1)}v^{(1)}\otimes t^{(1)}u^{(1)}$$

$$\otimes Q^{(2)}((Q_{-3}^{(1)})_1V_1^{(2)}\rightharpoonup U^{(2)})((Q_{-3}^{(1)})_2V_2^{(2)}\overline{T}_1^{(2)}\rightharpoonup$$

$$q^{(2)})(Q_{-2}^{(1)}V_3^{(2)}\overline{T}_2^{(2)}q_{-2}^{(1)}\widehat{V}^{(2)}\rightharpoonup u^{(2)})$$

$$\otimes Q_{-1}^{(1)}V_{-1}^{(1)}\overline{V}^{(2)}T^{(2)}q_{-1}^{(1)}\widehat{V}_{-1}^{(1)}v^{(2)}t^{(2)}\quad \text{by (C9), (QT3)}$$

$$=\sum Q_0^{(1)}(\overline{v}_0^{(1)}V^{(1)})_0\overline{V}^{(1)}\otimes T^{(1)}\overline{T}^{(1)}U^{(1)}\otimes q_0^{(1)}\widehat{V}_0^{(1)}v^{(1)}$$

$$\otimes t^{(1)}u^{(1)}\otimes Q^{(2)}(Q_{-3}^{(1)}\overline{v}^{(2)}\rightharpoonup U^{(2)})(Q_{-2}^{(1)}\overline{v}_{-1}^{(1)}V^{(2)}\overline{T}^{(2)}$$

$$\rightharpoonup q^{(2)}(q_{-2}^{(1)}\widehat{V}^{(2)}\rightharpoonup u^{(2)}))$$

$$\otimes Q_{-1}^{(1)}(\overline{v}_0^{(1)}V^{(1)})_{-1}\overline{V}^{(2)}T^{(2)}q_{-1}^{(1)}\widehat{V}_{-1}^{(1)}v^{(2)}t^{(2)}\quad \text{by (DC3)}'$$

$$=\sum Q_0^{(1)}\overline{v}_0^{(1)}V_0^{(1)}\overline{V}^{(1)}\otimes T^{(1)}\overline{T}^{(1)}U^{(1)}\otimes q_0^{(1)}\widehat{V}_0^{(1)}v^{(1)}$$

$$\otimes t^{(1)}u^{(1)}\otimes Q^{(2)}(Q_{-3}^{(1)}\overline{v}^{(2)}\rightharpoonup U^{(2)})(Q_{-2}^{(1)}\overline{v}_{-2}^{(1)}V^{(2)}\overline{T}^{(2)}$$

$$\rightharpoonup q^{(2)}(q_{-2}^{(1)}\widehat{V}^{(2)}\rightharpoonup u^{(2)}))$$

$$\otimes Q_{-1}^{(1)}\overline{v}_{-1}^{(1)}V_{-1}^{(1)}\overline{V}^{(2)}T^{(2)}q_{-1}^{(1)}\widehat{V}_{-1}^{(1)}v^{(2)}t^{(2)}$$

by comodule algebra

$$=\sum Q_0^{(1)}\overline{v}_0^{(1)}V^{(1)}\otimes T^{(1)}U^{(1)}\otimes q_0^{(1)}\widehat{V}_0^{(1)}v^{(1)}\otimes t^{(1)}u^{(1)}$$

$$\otimes Q^{(2)}(Q_{-3}^{(1)}\overline{v}^{(2)}\rightharpoonup U^{(2)})(Q_{-2}^{(1)}\overline{v}_{-2}^{(1)}V_1^{(2)}T_1^{(2)}\rightharpoonup$$

$$q^{(2)}(q_{-2}^{(1)}\widehat{V}^{(2)}\rightharpoonup u^{(2)}))$$

$$\otimes Q_{-1}^{(1)}\overline{v}_{-1}^{(1)}V_2^{(2)}T_2^{(2)}q_{-1}^{(1)}\widehat{V}_{-1}^{(1)}v^{(2)}t^{(2)}\quad \text{by (DC3)}', \text{(QT3)}$$

$$=(id\otimes\Delta)(R),$$

where $u=U$, $q=Q$, $V=\overline{V}=\widetilde{V}=\widehat{V}=v=\overline{v}$, $t=T=\overline{T}$.

So (QT2) is proved.

Similarly, we can prove (QT3) for R.

For any $b\otimes h\in B\otimes H$, we will check the condition (QT4) as follows:

$$R\Delta(b\otimes h)$$

$$=\sum Q_0^{(1)}V_0^{(1)}\overline{V}^{(1)}(T_1^{(1)}U_1^{(1)}\rightharpoonup b_1)$$

$$\otimes T_2^{(1)}U_2^{(1)}(b_2)_{-1}h_1\otimes Q^{(2)}(Q_{-2}^{(1)}V^{(2)}\rightharpoonup U^{(2)})$$

$$((Q_{-1}^{(1)}V_{-1}^{(1)}\overline{V}^{(2)}T^{(2)})_1\rightharpoonup (b_2)_0)\otimes (Q_{-1}^{(1)}V_{-1}^{(1)}\overline{V}^{(2)}T^{(2)})_2h_2$$

$$=\sum Q_0^{(1)}V_0^{(1)}\overline{V}^{(1)}(T^{(1)}U_1^{(1)}\rightharpoonup b_1)\otimes t^{(1)}U_2^{(1)}(b_2)_{-1}h_1$$

$$\otimes Q^{(2)}(Q_{-2}^{(1)}V^{(2)}\rightharpoonup U^{(2)})((Q_{-1}^{(1)}V_{-1}^{(1)}\overline{V}^{(2)}T^{(2)}t^{(2)})_1\rightharpoonup (b_2)_0)$$

$$\otimes (Q_{-1}^{(1)}V_{-1}^{(1)}\overline{V}^{(2)}T^{(2)}t^{(2)})_2h_2\quad \text{by (QT2)}$$

$$=\sum Q_0^{(1)}V_0^{(1)}\overline{V}^{(1)}(T^{(1)}U^{(1)}\rightharpoonup b_1)$$

$$\otimes t^{(1)}u^{(1)}(b_2)_{-1}h_1\otimes Q^{(2)}(Q_{-2}^{(1)}V^{(2)}\rightharpoonup U^{(2)}(T_1^{(2)}\rightharpoonup$$

$$u^{(2)}))((Q_{-1}^{(1)}V_{-1}^{(1)}\overline{V}^{(2)}T_2^{(2)}t^{(2)})_1\rightharpoonup (b_2)_0)$$

$$\otimes (Q_{-1}^{(1)}V_{-1}^{(1)}\overline{V}^{(2)}T_2^{(2)}t^{(2)})_2h_2\quad \text{by (SC2), (QT3)}$$

$$=\sum Q_0^{(1)}V^{(1)}(T^{(1)}U^{(1)}\rightharpoonup b_1)\otimes t^{(1)}u^{(1)}(b_2)_{-1}h_1$$

$$\otimes Q^{(2)}(Q_{-2}^{(1)}V_1^{(2)}\rightharpoonup U^{(2)}(T_1^{(2)}\rightharpoonup u^{(2)}))$$

$$((Q_{-1}^{(1)}V_2^{(2)}T_2^{(2)}t^{(2)})_1\rightharpoonup (b_2)_0)$$

$$\otimes (Q_{-1}^{(1)}V_2^{(2)}T_2^{(2)}t^{(2)})_2h_2\quad \text{by (DC3)}'$$

$$=\sum Q_0^{(1)}V^{(1)}(T^{(1)}U^{(1)}\rightharpoonup b_1)$$

$$\otimes t^{(1)}u^{(1)}(b_2)_{-1}h_1\otimes Q^{(2)}((Q_{-1}^{(1)})_1V_1^{(2)}\rightharpoonup U^{(2)}$$

$$(T_1^{(2)}\rightharpoonup u^{(2)}(t_1^{(2)}\rightharpoonup (b_2)_0)))$$

$$\otimes (Q_{-1}^{(1)})_2V_2^{(2)}T_2^{(2)}t_2^{(2)}h_2\quad \text{by module algebra}$$

$$=\sum Q_0^{(1)}V^{(1)}(T^{(1)}U^{(1)}\rightharpoonup b_1)$$

$$\otimes \overline{T}^{(1)}u^{(1)}h_1\otimes Q^{(2)}((Q_{-1}^{(1)})V_1^{(2)}\rightharpoonup U^{(2)}(T_1^{(2)}\rightharpoonup b_2u^{(2)}))$$

$$\otimes (Q_{-1}^{(1)})_2V_2^{(2)}T_2^{(2)}\overline{T}^{(2)}h_2\quad \text{by (QT3), (C2)}$$

$$=\sum V^{(1)}(t^{(1)}\rightharpoonup Q^{(1)})(T^{(1)}U^{(1)}\rightharpoonup b_1)\otimes \overline{T}^{(1)}u^{(1)}h_1$$

$$\otimes Q^{(2)}((V^{(2)}t^{(2)})_1\rightharpoonup U^{(2)}(T_1^{(2)}\rightharpoonup b_2u^{(2)}))$$

$$\otimes (V^{(2)}t^{(2)})_2T_2^{(2)}\overline{T}^{(2)}h_2\quad \text{by (C1)}$$

$$=\sum V^{(1)}(\widetilde{T}^{(1)}\rightharpoonup v^{(1)})(\overline{t}^{(1)}t^{(1)}\rightharpoonup Q^{(1)})(T^{(1)}\widetilde{t}^{(1)}U^{(1)}\rightharpoonup b_1)$$

$$\otimes \overline{T}^{(1)}u^{(1)}h_1\otimes Q^{(2)}((v^{(2)}t^{(2)})\rightharpoonup U^{(2)}(\widetilde{t}^{(2)}\rightharpoonup b_2u^{(2)}))$$

$$\otimes V^{(2)}\widetilde{T}^{(2)}\overline{t}^{(2)}T^{(2)}\overline{T}^{(2)}h_2\quad \text{by (DC3), (QT3)}$$

$$=\sum V^{(1)}(\widetilde{T}^{(1)}\rightharpoonup \overline{V}^{(1)}(\widehat{T}^{(1)}\rightharpoonup v^{(1)}))(\overline{t}^{(1)}t^{(1)}\rightharpoonup Q^{(1)})$$

$$(T^{(1)}\widetilde{t}^{(1)}U^{(1)}\rightharpoonup b_1)\otimes \overline{T}^{(1)}u^{(1)}h_1\otimes Q^{(2)}(v^{(2)}t_1^{(2)}\rightharpoonup$$

$$U^{(2)})(\overline{V}^{(2)}\widehat{T}^{(2)}t_2^{(2)}\widetilde{t}^{(2)}\rightharpoonup b_2u^{(2)})$$

$$\otimes V^{(2)}\widetilde{T}^{(2)}\overline{t}^{(2)}T^{(2)}\overline{T}^{(2)}h_2\quad \text{by (DC3)}$$

$$=\sum V^{(1)}(\widetilde{T}^{(1)}\rightharpoonup \overline{V}^{(1)}(\widehat{T}_1^{(1)}\rightharpoonup v^{(1)}))(\overline{t}^{(1)}\widehat{T}_2^{(1)}t^{(1)}\rightharpoonup Q^{(1)})$$

$$(T^{(1)}\widehat{T}_3^{(1)}U^{(1)}\rightharpoonup b_1)\otimes \overline{T}^{(1)}u^{(1)}h_1\otimes Q^{(2)}(v^{(2)}t^{(2)}\rightharpoonup$$

$$U^{(2)})(\overline{V}^{(2)}\widehat{T}^{(2)}\rightharpoonup b_2u^{(2)})\otimes V^{(2)}\widetilde{T}^{(2)}\overline{t}^{(2)}T^{(2)}\overline{T}^{(2)}h_2$$

by (QT3), (QT2)

$$=\sum V^{(1)}(\tilde{\overline{T}}^{(1)}\rightharpoonup\overline{V}^{(1)}(\tilde{t}^{(1)}\rightharpoonup\overline{v}^{(1)})(\hat{T}_1^{(1)}\rightharpoonup v^{(1)}))$$
$$(\overline{t}^{(1)}\hat{T}_2^{(1)}t^{(1)}\rightharpoonup Q^{(1)})(T^{(1)}\hat{T}_3^{(1)}U^{(1)}\rightharpoonup b_1)\otimes\overline{T}^{(1)}u^{(1)}h_1$$
$$\otimes Q^{(2)}(v^{(2)}t^{(2)}\rightharpoonup U^{(2)})(\overline{v}^{(2)}\hat{T}_1^{(2)}\rightharpoonup b_2)$$
$$(\overline{V}^{(2)}\tilde{t}^{(2)}\hat{T}_2^{(2)}\rightharpoonup u^{(2)})\otimes V^{(2)}\tilde{\overline{T}}^{(2)}\overline{t}^{(2)}T^{(2)}\overline{T}^{(2)}h_2$$

by (DC3)

$$=\sum V^{(1)}(T_1^{(1)}\rightharpoonup\overline{V}^{(1)}(\tilde{t}^{(1)}\rightharpoonup\overline{v}^{(1)})((\hat{T}^{(1)}\hat{t}^{(1)})_1\rightharpoonup v^{(1)}))$$
$$(T_2^{(1)}(\hat{T}^{(1)}\hat{t}^{(1)})_2\rightharpoonup(t^{(1)}\rightharpoonup Q^{(1)})(U^{(1)}\rightharpoonup b_1))$$
$$\otimes\overline{T}^{(1)}u^{(1)}h_1\otimes Q^{(2)}(v^{(2)}t^{(2)}\rightharpoonup U^{(2)})(\overline{v}^{(2)}\hat{t}^{(2)}\rightharpoonup b_2)$$
$$(\overline{V}^{(2)}\tilde{t}^{(2)}\hat{T}^{(2)}\rightharpoonup u^{(2)})\otimes V^{(2)}T^{(2)}\overline{T}^{(2)}h_2$$

by (QT3), (QT2)

$$=\sum V^{(1)}(T^{(1)}\rightharpoonup\overline{V}^{(1)}(\hat{T}^{(1)}\rightharpoonup(b_2)_0Q^{(1)}))$$
$$\otimes\overline{T}^{(1)}u^{(1)}h_1\otimes b_1((b_2)_{-1}\rightharpoonup Q^{(2)})(\overline{V}^{(2)}\hat{T}^{(2)}\rightharpoonup u^{(2)})$$
$$\otimes V^{(2)}T^{(2)}\overline{T}^{(2)}h_2 \quad \text{by (QT2), (WQT4)}'$$
$$=\sum(((b_2)_0Q^{(1)})_0\overline{V}^{(1)})_0V^{(1)}\otimes\overline{T}^{(1)}u^{(1)}h_1\otimes b_1((b_2)_{-1}\rightharpoonup$$
$$Q^{(2)})(((b_2)_0Q^{(1)})_{-1}\overline{V}^{(2)}\rightharpoonup u^{(2)})$$
$$\otimes(((b_2)_0Q^{(1)})_0\overline{V}^{(1)})_{-1}V^{(2)}\overline{T}^{(2)}h_2 \quad \text{by (C1)}$$
$$=\sum(b_2)_0Q_0^{(1)}\overline{V}_0^{(1)}V^{(1)}\otimes h_3\overline{T}^{(1)}u^{(1)}\otimes b_1((b_2)_{-3}\rightharpoonup Q^{(2)})$$
$$((b_2)_{-2}Q_{-2}^{(1)}\overline{V}^{(2)}h_1\rightharpoonup u^{(2)})\otimes(b_2)_{-1}Q_{-1}^{(1)}\overline{V}_{-1}^{(1)}V^{(2)}h_2\overline{T}^{(2)}$$

by (C4), (QT4)

$$=\sum(b_2)_0Q_0^{(1)}(h_2\rightharpoonup\overline{V}^{(1)})_0(h_4\rightharpoonup V^{(1)})$$
$$\otimes h_5\overline{T}^{(1)}u^{(1)}\otimes b_1((b_2)_{-3}\rightharpoonup Q^{(2)})((b_2)_{-2}Q_{-2}^{(1)}h_1\overline{V}^{(2)}$$
$$\rightharpoonup u^{(2)})\otimes(b_2)_{-1}Q_{-1}^{(1)}(h_2\rightharpoonup\overline{V}^{(1)})_{-1}h_3V^{(2)}\overline{T}^{(2)} \quad \text{by (C3)}$$
$$=\sum(b_2)_0(h_2\rightharpoonup Q^{(1)})_0(h_5\rightharpoonup\overline{V}_0^{(1)}V^{(1)})\otimes h_6\overline{T}^{(1)}u^{(1)}$$
$$\otimes b_1((b_2)_{-3}\rightharpoonup(h_1\rightharpoonup Q^{(2)}))((b_2)_{-2}(h_2\rightharpoonup Q^{(1)})_{-2}h_3\overline{V}^{(2)}$$
$$\rightharpoonup u^{(2)})\otimes(b_2)_{-1}(h_2\rightharpoonup Q^{(1)})_{-1}h_4\overline{V}_{-1}^{(1)}V^{(2)}\overline{T}^{(2)}$$

by (ii), (C5)

$$=\sum(b_2)_0(h_3\rightharpoonup Q_0^{(1)})(h_4\rightharpoonup\overline{V}_0^{(1)}V^{(1)})\otimes h_5\overline{T}^{(1)}u^{(1)}$$
$$\otimes b_1((b_2)_{-3}\rightharpoonup(h_1\rightharpoonup Q^{(2)}))((b_2)_{-2}(h_2Q_{-1}^{(1)})_1\overline{V}^{(2)}\rightharpoonup$$
$$u^{(2)})\otimes(b_2)_{-1}(h_2Q_{-1}^{(1)})_2\overline{V}_{-1}^{(1)}V^{(2)}\overline{T}^{(2)} \quad \text{by (ii)}$$
$$=\Delta^{\mathrm{cop}}(b\otimes h)R.$$

So (QT4) is proved. Thus, ($B*H$, R) is a quasitriagular Hopf algebra.

These complete the proof of Proposition 6.

Thus it follows from Propositions 4~6 we have:

Theorem 1 The biproduct Hopf algebra $B*H$ is quasitriangular if and only if there exist elements $T=\sum T^{(1)}\otimes T^{(2)}\in H\otimes H$, $Q=\sum Q^{(1)}\otimes Q^{(2)}\in B\otimes B$, $V=\sum V^{(1)}\otimes V^{(2)}\in B\otimes H$, $U=\sum U^{(1)}\otimes U^{(2)}\in H\otimes B$, such that ($H$, T) is a quasitriangular Hopf algebra, (B, H) is a compatible pair associated to (T, V), (B, H) is a skew compatible pair associated to (T, U), (B, Q) is a weak quasitriangular pair associated to (T, V, U) and the conditions C1~C9 are satisfied. Moreover, the quasitriangular structure R has a unique decomposition:

$$R=\sum Q_0^{(1)}V_0^{(1)}\overline{V}^{(1)}\otimes T^{(1)}U^{(1)}\otimes Q^{(2)}(Q_{-2}^{(1)}V^{(2)}\rightharpoonup U^{(2)})$$
$$\otimes Q_{-1}^{(1)}V_{-1}^{(1)}\overline{V}^{(2)}T^{(2)}.$$

Remark 5 Let H be a Hopf algebra, and B be an arbitrary H-module algebra, H-comodule coalgebra. Theorem 1 shows that if H is not a quasitriangular Hopf algebra then the $B*H$ is not a quasitriangular Hopf algebra either.

Corollary 1 Let $B*H$ be a biproduct Hopf algebra and (H, T) be a quasitriangular Hopf algebra, then $B*H$ is quasitriangular if there exists $Q=\sum Q^{(1)}\otimes Q^{(2)}\in B\otimes B$ such that the following conditions hold:

$$\text{(WQT1)}''\ \sum\varepsilon(Q^{(1)})Q^{(2)}=\sum Q^{(1)}\varepsilon(Q^{(2)})=1,$$
$$\text{(WQT2)}''\ (\Delta\otimes id)(Q)=\sum Q_0^{(1)}\otimes q^{(1)}\otimes Q^{(2)}(Q_{-1}^{(1)}\rightharpoonup q^{(2)}),$$
$$\text{(WQT3)}''\ (id\otimes\Delta)(Q)=\sum Q^{(1)}q^{(1)}\otimes q^{(2)}\otimes Q^{(2)},$$
$$\text{(WQT4)}''\ \sum Q_0^{(1)}(T^{(1)}\rightharpoonup b_1)\otimes Q^{(2)}(Q_{-1}^{(1)}T^{(2)}\rightharpoonup b_2)$$
$$=\sum(b_2)_0Q^{(1)}\otimes b_1((b_2)_{-1}\rightharpoonup Q^{(2)}),$$
$$\text{(C1)}'\ \sum T^{(1)}\rightharpoonup b\otimes T^{(2)}=\sum b_0\otimes b_{-1},$$
$$\text{(C2)}'\ \sum T^{(1)}b_{-1}\otimes(T^{(2)}\rightharpoonup b_0)=1_H\otimes b,$$
$$\text{(C3)}'\ \sum Q^{(1)}\otimes Q^{(2)}\varepsilon(h)=\sum h_2\rightharpoonup Q^{(1)}\otimes h_1\rightharpoonup Q^{(2)},$$
$$\text{(C4)}'\ \sum Q_0^{(1)}\otimes Q_{-1}^{(2)}Q_{-1}^{(1)}\otimes Q_0^{(2)}=\sum Q^{(1)}\otimes 1_H\otimes Q^{(2)},$$
$$\text{(C5)}'\ \sum Q_{-1}^{(1)}\otimes Q_0^{(1)}\otimes Q^{(2)}=\sum T^{(1)}\otimes Q^{(1)}\otimes T^{(2)}\rightharpoonup Q^{(2)}.$$

In this case, the quasitriangular structure of $B*H$ is $R=\sum Q_0^{(1)}\otimes T^{(1)}\otimes Q^{(2)}\otimes Q_{-1}^{(1)}T^{(2)}$.

Proof Letting $V=1_B\otimes 1_H\in B\otimes H$, $U=1_H\otimes 1_B\in H\otimes B$ in Theorem 1, we obtain this corollary.

Example 3 Let (H, T) be a triangular Hopf algebra and $(B, \overline{Q})$ be a quasitriangular Hopf algebra in ${}_H Mod$, then $B*H$ is quasitriangular.

Proof Let $Q=\sum T^{(1)} \rightharpoonup \overline{Q}^{(1)} \otimes T^{(2)} \rightharpoonup \overline{Q}^{(2)}$. Then the conditions in Corollary 1 are easily checked.

Corollary 2 Let $B*H$ be a biproduct Hopf algebra and (B, Q) be a quasitriangular Hopf algebra. Assume that H is cocommutative, then $R=\sum Q^{(1)}\otimes 1_H\otimes Q^{(2)}\otimes 1_H$ is a quasitriangular structure over $B*H$ if and only if:

$$(\text{C1})'' \quad b\otimes 1_H = \sum b_0 \otimes b_{-1},$$

$$(\text{C2})'' \quad \sum Q^{(1)}\otimes Q^{(2)}\varepsilon(h) = \sum h_2 \rightharpoonup Q^{(2)}\otimes h_1 \rightharpoonup Q^{(2)}.$$

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