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L^p-estimates on a ratio involving a Bessel process*

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Abstract: Let $Z=(Z_t)_{t\geq 0}$ be a Bessel process of dimension δ ($\delta>0$) starting at zero and let $K(t)$ be a differentiable function on $[0, \infty)$ with $K(t)>0$ ($\forall t\geq 0$). Then we establish the relationship between L^p-norm of $\log^{1/2}(1+\delta J_\tau)$ and L^p-norm of $\sup Z_t[t+k(t)]^{-1/2}$ ($0\leq t\leq\tau$) for all stopping times τ and all $0<p<+\infty$. As an interesting example, we show that $\|\log^{1/2}(1+\delta L_{m+1}(\tau))\|_p$ and $\|\sup Z_t\prod[1+L_j(t)]^{-1/2}\|_p$ ($0\leq j\leq m, j\in\mathbb{Z}$; $0\leq t\leq\tau$) are equivalent with $0<p<+\infty$ for all stopping times τ and all integer numbers m, where the function $L_m(t)$ ($t\geq 0$) is inductively defined by $L_{m+1}(t)=\log[1+L_m(t)]$ with $L_0(t)=1$.

Key words: Bessel processes, Diffusion process, Itô's formula, Domination relation
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INTRODUCTION

Throughout this paper, we shall work with a filtered complete probability space $(\Omega,F,(F_t),P)$ satisfying the usual conditions. Let $B=(B_t)_{t\geq 0}$ be a standard Brownian motion with $B_0=0$. Denote by $\mathbb{R}_+$ the set of all non-negative real numbers.

Recall that a diffusion process X starting at $x\geq 0$ is called the square of a Bessel process of dimension $\delta>0$ if

$$\mathrm{d}X_t=\delta\,\mathrm{d}t+2\sqrt{|X_t|}\,\mathrm{d}B_t,\quad X_0=x, \tag{1}$$

Clearly, this equation has a unique non-negative strong solution X, i.e., such that, for each $t\geq 0$ random variable X_t is $F_t^B=\sigma(B_s,\ 0\leq s\leq t)$-measurable. The process X is called the square of a Bessel process of dimension $\delta>0$ (in symbol, $X\in BESQ^\delta(x)$) (Revuz and Yor, 1998). The expression $r=\delta/2-1$ is called the index of the process. The process $Z=\sqrt{X_t}$ ($X\in BESQ^\delta(x)$) is called a Bessel process of dimension $\delta>0$ starting at $\sqrt{x}$. The Bessel process Z of dimension $\delta>0$ is a continuous non-negative Markovian process. The Bessel processes of dimension $\delta\geq 1$ are submartingales, and the Bessel processes of dimension $0<\delta<1$ are not semimartingales. See (Revuz and Yor, 1998) for Bessel processes with non-negative dimension. Furthermore, we can extend Bessel processes of dimension $\delta>0$ to $\delta<0$ (Dubins *et al.*, 1993; Göing-Jaeschke and Yor, 2003).

The main aim of this paper is to present an L^p ($0<p<+\infty$) estimate on the ratio of the form $\sup Z_t[t+k(t)]^{-1/2}$ ($0\leq t\leq\tau$) for all stopping times τ, where Z is a Bessel process of dimension $\delta>0$ starting at zero and $t\mapsto K(t)$ is a differentiable function on $\mathbb{R}_+$ with $K(t)>0$ ($\forall t\geq 0$). Our fundamental theorem is Theorem 1, where, for $X\in BESQ^\delta(0)$ with $\delta>0$ we show that the inequalities

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$$\frac{1}{b_p}\left\|\log\left(1+\delta\int_0^{\tau}\frac{\mathrm{d}t}{t+K(t)}\right)\right\|_p\leq\left\|\sup_{0\leq t\leq\tau}\frac{X_t}{t+K(t)}\right\|_p$$
$$\leq 4b_p\cdot 2^{\delta/2}\left\|\log\left(1+\delta\int_0^{\tau}\frac{\mathrm{d}t}{t+K(t)}\right)\right\|_p \quad (2)$$

hold for all stopping times τ and all $0<p<+\infty$, where $b_p=9(\mathrm{e}+2\mathrm{e}p)^{(1+2p)/p}$. As an interesting example, for every stopping time τ and every non-negative integer number m we have

$$\frac{1}{\sqrt{2b_{p/2}}}\left\|\log^{1/2}(1+\delta L_{m+1}(\tau))\right\|_p\leq\left\|\sup_{0\leq t\leq\tau}Z_t\Big/\sqrt{\prod_{j=0}^{m}(1+L_j(t))}\right\|_p$$
$$\leq\sqrt{8b_{p/2}}\,2^{\delta/4}\left\|\log^{1/2}(1+\delta L_{m+1}(\tau))\right\|_p,$$

with $0<p<+\infty$, where the function $t\mapsto L_{m+1}(t)$ on $\mathbb{R}_+$ is inductively defined by

$$L_{m+1}(t)=\log(1+L_m(t)),\quad m\geq 0,$$

with $L_0(t)=1$ (Graversen and Peskir, 2000; Yan, 2003; Yan and Zhu, 2004; 2005; Yan and Ling, 2005).

Finally, as an extension to inequalities (2), we consider the L^p estimate on the solution of the equation

$$\mathrm{d}X_t=(\delta f^2(t)-h(t)X_t)\mathrm{d}t+2f(t)\sqrt{|X_t|}\mathrm{d}B_t,\quad X_0=x,$$

where $\delta>0$ and f, h:$\mathbb{R}_+\to\mathbb{R}_+$ two continuous functions with $f(t)>0$ ($\forall t\geq 0$).

RESULT AND PROOF

In this section we shall give the proof of inequalities (2) and some related inequalities. Let $t\mapsto K(t)$ be a differentiable function on $\mathbb{R}_+$ with $K(t)>0$ ($\forall t\geq 0$) and let $\delta>0$. Assume that a: $\mathbb{R}_+\to\mathbb{R}_+$ is the solution to the equation

$$\frac{\mathrm{d}a}{\mathrm{d}t}-\frac{K'(t)}{K(t)}a=-\frac{2a^2}{K(t)},\quad a(0)=1, \quad (3)$$

and that

$$G(x)=\frac{1}{2}\int_0^x t^{-\delta/2}\mathrm{e}^{t/2}\,\mathrm{d}t\int_0^t s^{\delta/2-1}\mathrm{e}^{-s/2}\,\mathrm{d}s,\quad x\geq 0. \quad (4)$$

Define the function $(t,x)\mapsto F(t,x)$ by $F(t,x)=G(a(t)x)=1$. Then

$$\frac{\partial F}{\partial t}+\frac{\delta-K'(t)x}{K(t)}\frac{\partial F}{\partial x}+\frac{2x}{K(t)}\frac{\partial^2 F}{\partial x^2}=\frac{a(t)}{K(t)}, \quad (5)$$

and $F(t,0)=0$.

On the other hand, it is not difficult to check that the inequalities

$$2^{2-\delta/2}(\mathrm{e}^{x/4}-1)/\delta\leq G(x)\leq 2(\mathrm{e}^{x/2}-1)/\delta \quad (6)$$

hold $\forall x\geq 0$. Clearly, the upper bound in inequalities (6) is optimal, since $\lim_{x\to 0}\{G(x)/[2(\mathrm{e}^{x/2}-1)/\delta]\}=1$.

The lower bound in inequalities (6) may be replaced by

$$\frac{2}{\delta}\frac{\varepsilon^{\delta/2}}{1-\varepsilon}\left[\mathrm{e}^{(1-\varepsilon)x/2}-1\right],$$

with a fixed constant $\varepsilon\in(0,1)$.

Now, for $x\geq 0$ we define the function H_p:$\mathbb{R}_+\to\mathbb{R}_+$ by

$$H_p(G(x))=x^p,\quad p>0.$$

Then H_p is an increasing continuous function on $\mathbb{R}_+$ with $H_p(0)=0$ for every $0<p<+\infty$. For $x\geq 0$ we set

$$\tilde{H}_p=x\int_x^{+\infty}\frac{1}{s}\mathrm{d}H_p(s)+2H_p(x).$$

Lemma 1 Let H_p and $\tilde{H}_p$ be defined as above. Then for all $0<p<+\infty$ we have

$$\log^p(1+\delta x)\leq H_p(x)\leq 2^{p(2+\delta/2)}\log^p(1+\delta x),\ x\geq 0, \quad (7)$$

and for $0<p<1$ we have

$$\tilde{H}_p(x)\leq\frac{2-p}{1-p}H_p,\quad x\geq 0. \quad (8)$$

Proof The inequalities (7) follow from (6). This implies that the function $\tilde{H}_p$ is well defined for all $0<p<+\infty$.

To prove inequality (8), it is now enough to assume that

$$H_p(x)=A\log^p(1+\delta x),$$

with a constant A. For $x\geq 0$ we set

$$G_p(x)=\frac{x}{H_p(x)}\int_x^{\infty}\frac{1}{s}\,\mathrm{d}H_p(s).$$

An elementary calculation can show that for all $x\geq 0$ and all $0<p<1$

$$\lim_{x\to 0}G_p(x)=p/(1-p),\quad \lim_{x\to+\infty}G_p(x)=0,$$

and

$$0\leq G_p(x)\leq p/(1-p).$$

It follows that $\tilde{H}_p(x)\leq\frac{2-p}{1-p}H_p$ for all $x\geq 0$ and all $0<p<1$. This completes the proof.

Lemma 2 Let $D=(D_t)_{t\geq 0}$ be a non-negative right-continuous process, and let $A=(A_t)_{t\geq 0}$ be an increasing continuous process with $A_0=0$. Assume H: $\mathbb{R}_+\rightarrow\mathbb{R}_+$ is an increasing continuous function with $H(0)=0$. If for all bounded stopping times τ

$$E[D_\tau]\leq E[A_\tau],$$

then

$$E\left[\sup_{0\leq t\leq\tau}H(D_t)\right]\leq E[\tilde{H}(A_\tau)]$$

holds for all stopping times τ, where $\tilde{H}:\mathbb{R}_+\rightarrow\mathbb{R}_+$ is defined by

$$\tilde{H}(x)=x\int_x^{\infty}\frac{1}{s}\,\mathrm{d}H(s)+2H(x),\quad x\geq 0.$$

The proof of Lemma 2 can be found in (Revuz and Yor, 1998; Graversen and Peskir, 2000). The following lemma is a modification of Lemma 1 (Lenglart *et al.*, 1980), and it is a useful technique to obtain the L^p estimates of random variables (Barlow and Yor, 1981; Jacka and Yor, 1993).

Lemma 3 Let A and B be two continuous, (F_t)-adapted, increasing processes, with $A_0=0$ and $B_0=0$. Assume that there exist two constants α, $\beta>0$ such that

$$E[(A_T^\beta-A_S^\beta)^\alpha]\leq\|B_T\|_\infty^{\alpha\beta}\,P(S<T)$$

holds for all couples (S,T) of stopping times S, T with $S<T$. Then for any $0<p<+\infty$, we have

$$\|A_\infty\|_p\leq C_{p,\alpha,\beta}\|B_\infty\|_p,$$

where $C_{p,\alpha,\beta}=[\mathrm{e}+\mathrm{e}p/(\alpha\beta)]^{(1+p/\beta)/p}$.

Theorem 1 Let $X\in BESQ^\delta(x)$ with $\delta>0$ and let $t\mapsto K(t)$ be a differentiable function on $\mathbb{R}_+$ with $K(t)\geq 0$ $(\forall t\geq 0)$. Then the inequalities

$$\frac{1}{b_p}\|\log(1+\delta J_\tau)\|_p\leq\left\|\sup_{0\leq t\leq\tau}\frac{X_t}{t+K(t)}\right\|_p\leq 4b_p2^{\delta/2}\|\log(1+\delta J_\tau)\|_p \tag{9}$$

hold for all stopping times τ and all $0<p<+\infty$, where $J_\tau=\int_0^\tau\frac{\mathrm{d}t}{t+K(t)}$ and $b_p=9(\mathrm{e}+2\mathrm{e}p)^{(1+2p)/p}$.

Proof Set $U_t=X_t/K(t)$, $t\geq 0$. Then, by Itô's formula we have

$$\mathrm{d}U_t=2\frac{\sqrt{U_t}}{\sqrt{K(t)}}\mathrm{d}B_t+\frac{\delta-K'(t)U_t}{K(t)}\mathrm{d}t,$$

with $U_0=0$. Let a and G be given by Eqs.(3) and (4), respectively, and let $F(t,x)=G(a(t)x)$ for $t\geq 0$, $x\geq 0$. From Itô's formula and Eq.(5) it follows that

$$\begin{aligned}G(a(t)X)&=F(t,U_t)\\&=\int_0^t\frac{\partial}{\partial s}F(s,U_s)\mathrm{d}s+\int_0^t\frac{\delta-K'(s)}{K(s)}\frac{\partial}{\partial x}F(s,U_s)\mathrm{d}s\\&\quad+2\int_0^t\left(\frac{U_s}{K(s)}\right)^{1/2}\frac{\partial}{\partial x}F(s,U_s)\mathrm{d}B_s\\&\quad+2\int_0^t\frac{U_s}{K(s)}\frac{\partial^2}{\partial^2 x}F(s,U_s)\mathrm{d}s\\&=2\int_0^t\left(\frac{U_s}{K(s)}\right)^{1/2}\frac{\partial}{\partial x}F(s,U_s)\mathrm{d}B_s+\int_0^t\frac{a(s)}{K(s)}\mathrm{d}s.\end{aligned} \tag{10}$$

Noting that $a^{-1}(t)=1+t/K(t)$ by Eq.(3), we have $a(t)U_t=X_t/[t+K(t)]$ and $a(t)/K(t)=1/[t+K(t)]$ $(\forall t\geq 0)$. Combining this with Eq.(10), we find, for all bounded stopping times τ

$$E[F(\tau,X_\tau)]\equiv E\left[G\left(\frac{X_\tau}{\tau+K(\tau)}\right)\right]=E\left[\int_0^\tau\frac{\mathrm{d}t}{t+K(t)}\right]. \tag{11}$$

Now, for these processes $D_t=F(t,X_t)$ and $A_t=$

$\int_0^t \frac{\mathrm{d}s}{s+K(s)}$, by Lemma 1 and Lemma 2 we get

$$E\left[\sup_{0\le t\le\tau}\left(\frac{X_t}{t+K(t)}\right)^p\right]\equiv E\left[\sup_{0\le t\le\tau} H_p\left(F(t,X_t)\right)\right]$$
$$=E\left[\tilde{H}_p\left(\int_0^\tau \frac{\mathrm{d}t}{t+K(t)}\right)\right]\le\frac{2-p}{1-p}E\left[H_p\left(\int_0^\tau \frac{\mathrm{d}t}{t+K(t)}\right)\right]$$
$$\le\frac{2-p}{1-p}2^{p(2+\delta/2)}E\left[\log^p\left(1+\delta\int_0^\tau \frac{\mathrm{d}t}{t+K(t)}\right)\right]$$

for all stopping times τ and $0<p<1$. On the other hand, we see that Eq.(11) implies

$$E\left[\int_0^\tau \frac{\mathrm{d}t}{t+K(t)}\right]\le E\left[\sup_{0\le t\le\tau} F(t,K_t)\right]$$

for all bounded stopping times τ, and therefore by applying Lemma 1, Lemma 2 to these processes $D_t=\int_0^t \frac{\mathrm{d}s}{s+K(s)}$ and $A_t=\sup_{0\le s\le t} F(s,X_s)$, we get

$$E\left[H\left(\int_0^\tau \frac{\mathrm{d}t}{t+K(t)}\right)\right]\equiv E\left[\tilde{H}_p\left(\sup_{0\le t\le\tau} F(t,X_t)\right)\right]$$
$$\le\frac{2-p}{1-p}E\left[H_p\left(\sup_{0\le t\le\tau} F(t,X_t)\right)\right]$$
$$\le\frac{2-p}{1-p}E\left[\sup_{0\le t\le\tau}\left(\frac{X_t}{t+K(t)}\right)^p\right]$$

for all stopping times τ and all $0<p<1$. Thus, for $0<p<1$ we obtain the inequalities

$$\frac{1-p}{2-p}E[\log^p(1+\delta J_\tau)]\le E\left[\left(\sup_{0\le t\le\tau}\frac{X_t}{t+K(t)}\right)^p\right]$$
$$\le\frac{2-p}{1-p}2^{p(2+\delta/2)}E[\log^p(1+\delta J_\tau)]. \tag{12}$$

Next, we extend inequalities (12) to all $0<p<+\infty$ by Lemma 3. Consider any couple (S,T) of stopping times S, T with $S\le T$. Then, from the first inequality in (12) with $p=1/2$ and the inequality $\log(1+x)-\log(1+y)\le\log(1+x-y)$, $0\le x\le y$. We find

$$E\left(\sqrt{\log(1+\delta J_T)}-\sqrt{\log(1+\delta J_S)}\right)$$
$$\le E\left(\sqrt{\log[1+\delta(J_T-J_S)]}\right)$$
$$\le E\left[\sqrt{\ln\left(1+\delta J_{T1_{\{T>S\}}}\right)}\right]\le 3E\left[\sqrt{\sup_{0\le t\le T1_{\{T>S\}}}\frac{X_t}{t+K(t)}}\right]$$
$$\le 9\left\|\sup_{0\le t\le T}\frac{X_t}{t+K(t)}\right\|_\infty^{1/2} P(T>S), \tag{13}$$

where 1_A stands for the indicate function of set A. It follows from Lemma 3 with $\alpha=1$ and $\beta=1/2$ that

$$\left\|\log(1+\delta J_\tau)\right\|_p\le 9(\mathrm{e}+2\mathrm{e}p)^{\frac{1+2p}{p}}\left\|\sup_{0\le t\le\tau}\frac{X_t}{t+K(t)}\right\|_p$$

for all stopping times τ and all $0<p<+\infty$. To prove the left inequality in (9), for any couple (S,T) of stopping times S, T with $S\le T$, we have by the second inequality in (12) with $p=1/2$

$$E\left[\sqrt{\sup_{0\le t\le T}\frac{X_t}{t+K(t)}}-\sqrt{\sup_{0\le t\le S}\frac{X_t}{t+K(t)}}\right]$$
$$\le\sqrt{\sup_{S\le t\le T}\left|\frac{X_t}{t+K(t)}-\frac{X_S}{S+K(S)}\right|}$$
$$\le E\left[\sqrt{\sup_{0\le t\le(T-S)1_{\{S<T\}}}\frac{X_{t+S}}{t+S+K(t+S)}}\right]$$
$$\le 2^{\delta/4}\cdot 6E\left(\sqrt{\log(1+\delta J_\tau)}1_{\{T>S\}}\right)$$
$$\le\left\|2^{\delta/4}\cdot 36\log(1+\delta J_\tau)\right\|_\infty^{1/2} P(S<T),$$

which shows for all stopping times τ and all $0<p<+\infty$,

$$\left\|\sup_{0\le t\le\tau}\frac{X_t}{t+K(t)}\right\|_p\le 2^{\delta/2}\cdot 36(\mathrm{e}+2\mathrm{e}p)^{(1+2p)/p}\left\|\log(1+\delta J_\tau)\right\|_p$$

by Lemma 3 with $\alpha=1$ and $\beta=1/2$. This completes the proof.

Corollary 1 Let Z be a Bessel process of dimension $\delta>0$ starting at zero and let $t\mapsto K(t)$ be a differentiable function on $\mathbb{R}_+$ with $K(t)>0$ ($\forall t\ge 0$). Then the inequalities

$$\frac{1}{a_p}\left\|\log^{1/2}(1+\delta J_\tau)\right\|_p\le\left\|\sup_{0\le t\le\tau}\frac{Z_t}{\sqrt{t+K(t)}}\right\|_p$$
$$\le 2a_p\cdot 2^{\delta/4}\left\|\log^{1/2}(1+\delta J_\tau)\right\|_p$$

hold for all stopping times τ and all $0<p<+\infty$, where $a_p=3(\mathrm{e}+\mathrm{e}p)^{(1+p)/p}$.

Corollary 2 Let Z be a Bessel process of dimension $\delta>0$ starting at zero and let $0<p<+\infty$. For every non-negative integer number m we define the function $t\mapsto L_{m+1}(t)$ on $\mathbb{R}_+$ inductively by $L_{m+1}(t)=\log[1+L_m(t)]$ with $L_0(t)=t$. Then the inequalities

$$\frac{1}{\sqrt{2}a_p}\left\|\sqrt{\log[1+\delta L_{m+1}(\tau)]}\right\|_p\le\left\|\sup_{0\le t\le\tau}Z_t\Big/\sqrt{\prod_{j=0}^{m}[1+L_j(t)]}\right\|_p$$
$$\le 2^{3/2+\delta/4}a_p\left\|\sqrt{\log[1+\delta L_{m+1}(\tau)]}\right\|_p$$

hold for all stopping times τ.

Proof Corollary 2 follows from Corollary 1 by taking $K(t)=\prod_{j=0}^{m}(1+L_j(t))$, and some simple estimates.

Corollary 3 Let $r>1/2$ and let Z be a Bessel process of dimension $\delta>0$ starting at zero. Then we have

$$\left\|\log^{1/2}\left[1+\frac{\delta}{2r-1}\left(1-\frac{1}{(1+\tau)^{2r-1}}\right)\right]\right\|_p\le\sqrt{2}a_p\left\|\sup_{0\le t\le\tau}\frac{Z_t}{(1+t)^r}\right\|_p,$$
$$\left\|\sup_{0\le t\le\tau}\frac{Z_t}{(1+t)^r}\right\|_p\le 2^{3/2+\delta/4}a_p\left\|\log^{1/2}\left[1+\frac{\delta}{2r-1}\left(1-\frac{1}{(1+\tau)^{2r-1}}\right)\right]\right\|_p,$$

for all $0<p<+\infty$ and all stopping times τ.

Proof Take $K(t)=(1+t)^{2r}$, $r>1/2$. Then we have $\forall t\ge 0$

$$\frac{1}{\sqrt{2}}\frac{Z_t}{(1+t)^r}\le\frac{Z_t}{\sqrt{t+K(t)}}\le\frac{Z_t}{(1+t)^r},$$

and

$$\frac{1}{2(2r-1)}\left(1-\frac{1}{(1+t)^{2r}}\right)\le\int_0^t\frac{\mathrm{d}s}{s+K(s)}\le\frac{1}{2r-1}\left(1-\frac{1}{(1+t)^{2r}}\right).$$

Thus, the corollary follows from Corollary 1.

From Corollary 3, we see that

$$E\left[\sup_{0\le t\le\infty}\frac{(Z_t)^p}{(1+t)^{rp}}\right]\sim\frac{\delta^p}{(2r-1)^p}$$

for all $0<p<+\infty$ as $r\to\infty$.

Corollary 4 Let Z be a Bessel process of dimension $\delta>0$ starting at zero. Then we have for all $0<p<+\infty$ and all stopping times τ

$$\frac{1}{\sqrt{2}a_p}\left\|\sqrt{\log[1+\delta(1-\mathrm{e}^{-\tau})]}\right\|_p\le\left\|\sup_{0\le t\le\tau}\mathrm{e}^{-t/2}Z_t\right\|_p$$
$$\le 2^{3/2+\delta/4}a_p\left\|\sqrt{\log[1+\delta(1-\mathrm{e}^{-\tau})]}\right\|_p,$$

in particular, for all $0<p<+\infty$ we have

$$\frac{1}{\sqrt{2}a_p}\sqrt{\log(1+\delta)}\le\left\|\sup_{0\le t<\infty}\mathrm{e}^{-t/2}Z_t\right\|_p$$
$$\le 2^{3/2+\delta/4}a_p\sqrt{\log(1+\delta)}.$$

Proof Corollary 4 follows from Corollary 1 by taking $K(t)=\mathrm{e}^t-t$, $t\ge 0$.

From Corollary 4, we see that

$$\frac{1}{12\sqrt{2}\mathrm{e}^2}\le E\left[\sup_{0\le t<\infty}\frac{\mathrm{e}^{-t/2}Z_t}{\sqrt{\log(1+\delta)}}\right]\le 12\times 2^{3/2+\delta/4}\mathrm{e}^2,$$

Remark 1 From these inequalities above, one can perhaps get some asymptotic estimates associated with some random variables as $\delta\to\infty$. However, we cannot settle this question so far.

Finally, as the end of this paper, we extend Lemma 4 to general diffusion processes given by the equation

$$\mathrm{d}X_t=(\delta f^2(t)-h(t)X_t)\,\mathrm{d}t+2f(t)\sqrt{|X_t|}\,\mathrm{d}B_t,\ X_0=x, \tag{14}$$

where $\delta>0$ and f, h: $\mathbb{R}_+\to\mathbb{R}_+$ two continuous functions with $0<a\le f(t)\le b<\infty$ ($\forall t\ge 0$). Clearly, Eq.(14) admits a unique solution and the solution is strong (Ikeda and Watanabe, 1981; Revuz and Yor, 1998; Rogers and Williams, 1987), we deduce the solution $X\ge 0$ ($\forall t\ge 0$). In the following discussion, we suppose $x=0$ for simplicity.

Let η:$\mathbb{R}_+\to\mathbb{R}_+$ be the solution of the equation

$$\mathrm{d}\eta/\mathrm{d}t-h(t)\eta(t)=-\eta^2f^2(t),\ \eta(0)=1, \tag{15}$$

and define F: $\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ by $F(t,x)=G(\eta(t)x)$, where G is given by Eq.(4). Then we have

$$\frac{\partial F}{\partial t}+(\delta f^2(t)-h(t)x)\frac{\partial F}{\partial x}+2f^2(t)x\frac{\partial^2 F}{\partial x^2}=\eta(t)f^2(t), \tag{16}$$

and $F(t,0)=0$, $\forall t\geq 0$. It follows from Itô's formula that

$$G(\eta(t)X_t)=F(t,X_t)$$
$$=2\int_0^t f(s)\sqrt{X_s}\frac{\partial}{\partial x}F(s,X_s)\,\mathrm{d}B_s+\int_0^t \eta(s)f^2(s)\,\mathrm{d}s.$$

Thus, proceeding as in the proof of Lemma 4 one can give the following theorem.

Theorem 2 Let the process X be given by Eq.(14) with $X_0=0$ and let $\eta:\mathbb{R}_+\rightarrow\mathbb{R}_+$ be the solution to Eq.(15). For $t\geq 0$ we define

$$J_t=\int_0^t \eta(s)f^2(s)\,\mathrm{d}s.$$

Then for all $0<p<+\infty$ and all stopping times τ, we have

$$\frac{1}{b_p}\|\log(1+\delta J_\tau)\|_p\leq\left\|\sup_{0\leq t\leq\tau}\eta(t)X_t\right\|_p \leq 2^{2+\delta/2}b_p\|\log(1+\delta J_\tau)\|_p, \quad (17)$$

where $b_p=9(\mathrm{e}+2\mathrm{e}p)^{(1+2p)/p}$, in particular, for $0<p<1$ we have

$$\alpha_p E\left[\log^p(1+\delta J_\tau)\right]\leq E\left[\sup_{0\leq t\leq\tau}(\eta(t)X_t)^p\right] \leq\frac{1}{\alpha_p}2^{p(2+\delta/2)}E[\log^p(1+\delta J_\tau)],$$

where $\alpha_p=(1-p)/(2-p)$.

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