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Precise asymptotics in the law of the logarithm for random fields in Hilbert space*

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Abstract: Consider the positive d-dimensional lattice $\mathbb{Z}_+^d$ ($d\geq2$) with partial ordering $\leq$, let $\{X_K; K\in\mathbb{Z}_+^d\}$ be i.i.d. random variables taking values in a real separable Hilbert space $(H, \|\cdot\|)$ with mean zero and covariance operator $\sum$, and set partial sums $S_N=\sum_{K\leq N}X_K$, $K, N\in\mathbb{Z}_+^d$. Under some moment conditions, we obtain the precise asymptotics of a kind of weighted infinite series for partial sums S_N as $\varepsilon\searrow0$ by using the truncation and approximation methods. The results are related to the convergence rates of the law of the logarithm in Hilbert space, and they also extend the results of (Gut and Spătaru, 2003).

Key words: The law of the logarithm, Random field, Hilbert space, Tail probability, Truncation method
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INTRODUCTION

Let $\{X, X_n; n\geq1\}$ be a sequence of i.i.d. random variables and $S_n=\sum_{k=1}^{n}X_k$, for $n\geq1$. We have the famous following result, for $0<p<2$ and $r\geq p$,

$$\sum_{n=1}^{\infty}n^{r/p-2}P(|S_n|\geq\varepsilon n^{1/p})<\infty,\ \varepsilon>0,$$

if and only if $E|X|^r<\infty$ and when $r\geq1$, $EX=0$. For $r=2$, $p=1$, the sufficiency was proved by Hsu and Robbins (1947), and the necessity by Erdös (1949; 1950). For the case $r=p=1$, we refer to (Spitzer, 1956); and for the general result to (Baum and Katz, 1965).

The sums obviously tend to infinity as $\varepsilon\searrow0$. It is interesting to find the exact rate at which this occurs. The first result following to this end was given by Heyde (1975):

$$\lim_{\varepsilon\searrow0}\varepsilon^2\sum_{n=1}^{\infty}P(|S_n|\geq\varepsilon n)=EX^2,$$

if and only if $EX=0$ and $EX^2<\infty$. Later, Chen (1978) and Gut and Spătaru (2000a) both studied the precise asymptotics of the infinite sums as $\varepsilon\searrow0$. Moreover, Gut and Spătaru (2000b; 2003) studied the precise asymptotics of the law of the iterated logarithm and the precise asymptotics for multidimensionally indexed random variables. Lanzinger and Stadtmüller (2004), Spătaru (2004a; 2004b) and Huang and Zhang (2005) established the precise rates in some different cases. The purpose of this paper is to establish the precise asymptotics for random fields in Hilbert space, which extend the results of (Gut and Spătaru, 2003) and (Huang and Zhang, 2005).

In this context, let $\mathbb{Z}_+^d$ ($d\geq2$) denote the positive d-dimensional lattice with partial ordering $\leq$. The notation $M\leq N$, where $M=(m_1,m_2,\ldots,m_d)$ and $N=(n_1,n_2,\ldots,n_d)$, thus means that $m_k\leq n_k$, for $k=1, 2, \ldots, d$. We also use $|N|$ for $\prod_{i=1}^{d}n_i$. Throughout the remainder of this paper, we assume that X and $\{X_K;$

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$K\in\mathbb{Z}_+^d\}$ are i.i.d. random variables taking values in a real separable Hilbert space (H, ||·||) with mean zero and covariance ∑. Denote the largest eigenvalue of ∑ by σ^2, i.e.

$$\sigma^2=\sup\{E[(X,y)^2]:\|y\|\le 1\},$$

where (· , ·) denotes the scalar product in H. Let l be the dimension of the corresponding eigenspace, and σ_i^2 ($1\le i\le l'$) the positive eigenvalues of ∑ arranged in a nonincreasing order and take into account the multiplicities. Further, if $l'<\infty$, put $\sigma_i^2=0$, $i\ge l'$. Note that we always have $\sigma_i^2=\sigma^2$, $1\le i\le l$, and $\sigma_i^2<\sigma^2$, $i>l$ (Einmahl, 1991). Set $S_N=\sum_{K\le N}X_K$, and $\log x=\ln(x\vee e)$. The following theorems are our main results:

Theorem 1 Let $b>-d$. Suppose that $EX=0$ and $E[\|X\|^2(\log\|X\|)^{b+d+3}]<\infty$. Then, we have

$$\lim_{\varepsilon\searrow 0}\varepsilon^{2(b+d)}\sum_N\frac{(\log|N|)^b}{|N|}P\left(\|S_N\|\ge\sigma\varepsilon\sqrt{|N|\log|N|}\right)=\frac{1}{(d-1)!}\frac{\sigma^{-2(b+d)}}{b+d}E[\|Y\|]^{2(b+d)},$$

where Y is a Gaussian random variable taking value in a real separable Hilbert space with mean zero and covariance operator ∑.

Now, let $\{X_n;\ n\ge 1\}$ be a sequence of i.i.d. random variables taking values in a real separable Hilbert space (H, ||·||) with mean zero and covariance ∑. The following theorem is much more interesting, which reflects the convergence rate of the law of the logarithm more directly.

Theorem 2 Suppose that $EX=0$ and $E[\|X\|^2(\log\|X\|)^{b+4}]<\infty$. For $-1<b<0$, we have that

$$\lim_{\varepsilon\searrow 0}\varepsilon^{2(b+1)}\sum_{n=1}^{\infty}\frac{(\log n)^b}{n}I\left(\|S_n\|\ge\sigma\varepsilon\sqrt{n\log n}\right)=\frac{\sigma^{-2(b+1)}}{b+1}E[\|Y\|]^{2(b+1)},\qquad L_2 \text{ and } a.s.$$

i.e.

$$\lim_{\varepsilon\searrow 0}\varepsilon^{2(b+1)}\sum_{n\in N(\varepsilon)}\frac{(\log n)^b}{n}=\frac{\sigma^{-2(b+1)}}{b+1}E[\|Y\|]^{2(b+1)},\ L_2 \text{ and } a.s.,$$

where $N(\varepsilon)=\{n:\|S_n\|\ge\sigma\varepsilon\sqrt{n\log n}\}$.

Conjecture We believe that Theorem 2 has a similar form for the random field case. To get such a result, we think a different approach is necessary.

The plan of this paper is as follows. Section 2 contains some preliminaries for random fields. The proofs of Theorem 1 and Theorem 2 are listed in Section 3 and Section 4, respectively. Let C, C', C_i and A etc. denote positive constants whose values possibly vary from place to place. The notation of $a_n\sim b_n$ means $a_n/b_n\to 1$ as $n\to\infty$.

PRELIMINARIES

A first important observation is that inequalities which do not depend on the (partial) order of the index set $\mathbb{Z}_+^d$, such as the triangle inequality, moment inequalities for sums, and so on, remain valid. Namely, such relations only depend on that, if $\{X_K,\ K\in\mathbb{Z}_+^d\}$ are random variables and $\{S_N;N\in\mathbb{Z}_+^d\}$ their partial sums, then S_N is simply a sum of $|N|$ random variables.

The following quantities and their asymptotic behavior turn out to be crucial. Let

$d(m)=Card\{K:|K|=m\}$ and $M(m)=Card\{K:|K|\le m\}$.

The following asymptotics hold (Hardy and Wright, 1954):

$$M(m)\sim\frac{m(\log m)^{d-1}}{(d-1)!}\quad\text{as } m\to\infty. \tag{1}$$

Another important observation is that, since all terms in the sums we consider are nonnegative, we may change the order of summation, in particular as follows

$$\sum_N\cdots=\sum_{m\ge 1}\sum_{|N|=m}\cdots \tag{2}$$

More important, whenever the functions involving N only depend on the value of $|N|$, the second summation can be simplified further. For example, for the sum in Theorem 1, we have

$$\sum_N \frac{(\log|N|)^b}{|N|} P\left(\|S_N\| \geq \sigma\varepsilon\sqrt{|N|\log|N|}\right)$$
$$= \sum_{m\geq 1}\sum_{|N|=m} \frac{(\log|N|)^b}{|N|} P\left(\|S_N\| \geq \sigma\varepsilon\sqrt{|N|\log|N|}\right)$$
$$= \sum_{m\geq 1} d(m)\frac{(\log m)^b}{m} P\left(\|S_m\| \geq \sigma\varepsilon\sqrt{m\log m}\right).$$

First we list a lemma useful in the proofs of our main results.

Lemma 1 Let a_n, b_n, $c_n>0$, $A_n=\sum_{k=1}^n a_k$ and $B_n=\sum_{k=1}^n b_k$. Suppose that $A_n \sim B_n$ and $\sum_{k=1}^n b_k c_k \to\infty$ as $n\to\infty$. Further, suppose one of the following conditions is satisfied:

(1) The sequence $\{c_n\}$ is eventually non-increasing;

(2) The sequence $\{c_n\}$ is eventually non-decreasing, and $\sum_{k=1}^n b_k c_k \approx B_n c_{n+1}$.

Then we have $\sum_{k=1}^n a_k c_k \sim \sum_{k=1}^n b_k c_k$ as $n\to\infty$.

Proof It follows from Lemma 2.3 in (Huang and Zhang, 2005).

PROOF OF THEOREM 1

In this section, we will first prove Theorem 1 in the case $\{X_K;\ K\in\mathbb{Z}_+^d\}$ are Gaussian random variables. And then, by using the truncation and approximation methods, we show that it still holds for the general case.

Now, let Y be a nondegenerate Gaussian random variable with mean zero and covariance operator $\sum$. The following is our result:

Proposition 1 For $b>-d$, we have

$$\lim_{\varepsilon\searrow 0}\varepsilon^{2(b+d)}\sum_N \frac{(\log|N|)^b}{|N|} P\left(\|Y\| \geq \sigma\varepsilon\sqrt{\log|N|}\right) = \frac{1}{(d-1)!}\frac{\sigma^{-2(b+d)}}{b+d}E[\|Y\|]^{2(b+d)}. \quad (3)$$

Proof Since $\frac{(\log m)^b}{m}$ is eventually non-increasing and $\sum_{m=1}^k d(m) \sim \frac{1}{(d-1)!}\sum_{m=1}^k (\log m)^{d-1}$ as $k\to\infty$ [Lemma 2.7 in (Gut and Spătaru, 2003)], it follows from Lemma 1 that, as $k\to\infty$

$$\sum_{m=1}^k \frac{d(m)(\log m)^b}{m} \sim \frac{1}{(d-1)!}\sum_{m=1}^k \frac{(\log m)^{b+d-1}}{m}.$$

It also means that for any $0<\delta<1$, there exists $k_0=k_0(\delta)$ such that for $k\geq k_0$

$$\frac{1-\delta}{(d-1)!}\sum_{m=1}^k \frac{(\log m)^{b+d-1}}{m} \leq \sum_{m=1}^k \frac{d(m)(\log m)^b}{m} \leq \frac{1+\delta}{(d-1)!}\sum_{m=1}^k \frac{(\log m)^{b+d-1}}{m}.$$

Hence, we have that

$$\sum_N \frac{(\log|N|)^b}{|N|} P\left(\|Y\| \geq \sigma\varepsilon\sqrt{\log|n|}\right)$$
$$= \sum_{m\geq 1}\frac{d(m)(\log m)^b}{m} P\left(\|Y\| \geq \sigma\varepsilon\sqrt{\log m}\right)$$
$$= \sum_{m\geq 1}\frac{d(m)(\log m)^b}{m}\cdot\sum_{k\geq m} P\left(\sigma\varepsilon\sqrt{\log k} \leq \|Y\| \leq \sigma\varepsilon\sqrt{\log(k+1)}\right)$$
$$= \sum_{k\geq 1}\sum_{m=1}^k \frac{d(m)(\log m)^b}{m}\cdot P\left(\sigma\varepsilon\sqrt{\log k} \leq \|Y\| \leq \sigma\varepsilon\sqrt{\log(k+1)}\right)$$
$$\leq C + \frac{1+\delta}{(d-1)!}\sum_{k\geq k_0}\sum_{m=1}^k\left[\frac{(\log m)^{b+d-1}}{m}\cdot P\left(\sigma\varepsilon\sqrt{\log k} \leq \|Y\| \leq \sigma\varepsilon\sqrt{\log(k+1)}\right)\right]$$
$$\leq C + \frac{1+\delta}{(d-1)!}\sum_{m=1}^{k_0}\frac{(\log m)^{b+d-1}}{m}\cdot\sum_{k\geq k_0} P\left(\sigma\varepsilon\sqrt{\log k} \leq \|Y\| \leq \sigma\varepsilon\sqrt{\log(k+1)}\right)$$
$$+\frac{1+\delta}{(d-1)!}\sum_{m\geq k_0}\frac{(\log m)^{b+d-1}}{m}\cdot\sum_{k\geq m} P\left(\sigma\varepsilon\sqrt{\log k} \leq \|Y\| \leq \sigma\varepsilon\sqrt{\log(k+1)}\right)$$
$$\leq C' + \frac{1+\delta}{(d-1)!}\sum_{m\geq k_0}\frac{(\log m)^{b+d-1}}{m} P\left(\|Y\| \geq \sigma\varepsilon\sqrt{\log m}\right)$$
$$= C' + \frac{1+\delta}{(d-1)!}\int_{k_0}^{\infty}\frac{(\log x)^{b+d-1}}{x} P\left(\|Y\| \geq \sigma\varepsilon\sqrt{\log x}\right)\mathrm{d}x$$
$$= C' + \frac{2(1+\delta)}{(d-1)!}(\sigma\varepsilon)^{-2(b+d)}\int_{\sigma\varepsilon\sqrt{\log k_0}}^{\infty} z^{2(b+d)-1}P(\|Y\| \geq z)\mathrm{d}z.$$

Thus, we have

$$\limsup_{\varepsilon\searrow 0}\varepsilon^{2(b+d)}\sum_N \frac{(\log|N|)^b}{|N|}P\left(\|Y\|\geq\sigma\varepsilon\sqrt{\log|N|}\right)$$
$$=\limsup_{\varepsilon\searrow 0}\varepsilon^{2(b+d)}\sum_{m\geq 1}d(m)\frac{(\log m)^b}{m}P\left(\|Y\|\geq\sigma\varepsilon\sqrt{\log m}\right)$$
$$\leq\limsup_{\varepsilon\searrow 0}\left\{C'\varepsilon^{2(b+d)}+\frac{2(1+\delta)}{(d-1)!}\sigma^{-2(b+d)}\cdot\int_{\sigma\varepsilon\sqrt{\log k_0}}^{\infty}z^{2(b+d)-1}P(\|Y\|\geq z)\mathrm{d}z\right\}$$
$$\leq\frac{1+\delta}{(d-1)!}\frac{\sigma^{-2(b+d)}}{b+d}E[\|Y\|]^{2(b+d)}.$$

Similarly, we have that

$$\liminf_{\varepsilon\searrow 0}\varepsilon^{2(b+d)}\sum_N \frac{(\log|N|)^b}{|N|}P\left(\|Y\|\geq\sigma\varepsilon\sqrt{\log|N|}\right)\geq\frac{1-\delta}{(d-1)!}\frac{\sigma^{-2(b+d)}}{b+d}E[\|Y\|]^{2(b+d)}.$$

By the arbitrariness of δ, we have proved the proposition.

Without loss of generality, we assume that σ=1 in the sequel. For each $m\geq 1$, and $1\leq j\leq m$, we define

$$X'_{mj}=X_jI\left(\|X_j\|\leq\frac{\sqrt{m}}{(\log m)^2}\right),$$
$$\bar{X}'_{mj}=X'_{mj}-E[X'_{mj}],$$
$$S'_{mj}=\sum_{i=1}^{j}X'_{mj},\ \bar{S}'_{mj}=\sum_{i=1}^{j}\bar{X}'_{mj}.$$

Under the moment assumption

$$E[\|X\|^2(\log\|X\|)^{b+d+3}]<\infty,\ b>-d,$$

it is easily seen that

$$\frac{E\|\bar{S}'_{mm}\|}{\sqrt{m\log m}}\leq\left(\frac{E\|\bar{S}'_{mm}\|^2}{m\log m}\right)^{1/2}\leq C\Big/\sqrt{\log m}\to 0.$$

Also denote

$$\varDelta_m=\|\bar{S}'_{mm}-S_m\|,\ q_m=P\left(\varDelta_m>\frac{\sqrt{m}}{(\log m)^2}\right),$$

and $$p_m=\left(\frac{\sqrt{m}}{(\log m)^2}\right)^{-3}\sum_{j=1}^{m}E[\|\bar{X}'_{mj}\|]^3.$$

Lemma 2 (Einmahl, 1991) Let ξ_1, ξ_2, …, ξ_n be independent H-valued random variables with $E\xi_i$=0, $E[\|\xi_i\|^3]<\infty$, and let Y_1, Y_2, …, Y_n be independent Gaussian mean zero random variables with $Cov(\xi_i)$= $Cov(Y_i)$, i=1, 2, …, n. Here $Cov(\xi_i)$ denotes the covariance operator of H-valued random variables. Then we have that, for any s, t>0,

$$P\left(\left\|\sum_{i=1}^n\xi_i\right\|\geq s\right)\leq P\left(\left\|\sum_{i=1}^nY_i\right\|\geq s-t\right)+Ct^{-3}\sum_{i=1}^nE[\|\xi_i\|^3],\tag{4}$$

and

$$P\left(\left\|\sum_{i=1}^n\xi_i\right\|\geq s\right)\geq P\left(\left\|\sum_{i=1}^nY_i\right\|\geq s+t\right)-Ct^{-3}\sum_{i=1}^nE[\|\xi_i\|^3],\tag{5}$$

where C is a universal constant.

Lemma 3 Let $b>-d$. Suppose that EX=0 and $E[\|X\|^2(\log\|X\|)^{b+d+3}]<\infty$. Then we have

$$\sum_{m\geq 1}\frac{d(m)(\log m)^b}{m}p_m<CE[\|X\|^2(\log\|X\|)^{b+d+3}]<\infty.$$

Proof It is easy to check that $(\log m)^b p_m/m$ is eventually non-increasing. Thus it follows from Lemma 1 that

$$\sum_{m\geq 1}\frac{d(m)(\log m)^b}{m}p_m<C_1+\frac{1+\delta}{(d-1)!}\sum_{m\geq k_0}\frac{(\log m)^{b+d-1}}{m}p_m$$
$$\leq C_1+\frac{1+\delta}{(d-1)!}\sum_{m\geq k_0}m^{-3/2}(\log m)^{b+d+5}\cdot\sum_{j=1}^{m}E\left[\|X\|^3I\left(\frac{\sqrt{j-1}}{(\log(j-1))^2}<\|X\|\leq\frac{\sqrt{j}}{(\log j)^2}\right)\right]$$
$$\leq C_1+\frac{1+\delta}{(d-1)!}\sum_{j=1}^{\infty}E\left[\|X\|^3I\left(\frac{\sqrt{j-1}}{(\log(j-1))^2}<\|X\|\leq\frac{\sqrt{j}}{(\log j)^2}\right)\right]\sum_{m\geq j}m^{-3/2}(\log m)^{b+d+5}$$
$$\leq C_1+C_2\frac{1+\delta}{(d-1)!}\sum_{j=1}^{\infty}E\left[\|X\|^3I\left(\frac{\sqrt{j-1}}{(\log(j-1))^2}<\|X\|\leq\frac{\sqrt{j}}{(\log j)^2}\right)\right]j^{-1/2}(\log j)^{b+d+5}$$
$$\leq CE[\|X\|^2(\log\|X\|)^{b+d+3}]<\infty.$$

Lemma 4 Let $b>-d$. Suppose that $EX=0$ and $E[\|X\|^2(\log\|X\|)^{b+d+3}]<\infty$. Then we have

$$\sum_{m\ge1}\frac{d(m)(\log m)^b}{m}q_m<CE[\|X\|^2(\log\|X\|)^{b+d+3}]<\infty.$$

Proof For $m\ge1$, let

$$T_m=mE\left[\|X\|I\left(\|X\|>\frac{\sqrt{m}}{(\log m)^2}\right)\right].$$

Then we have that $\left\|E\left(\sum_{i=1}^{j}X'_{mi}\right)\right\|\le T_m$, for $1\le j\le m$. Denote $T=\left\{m:T_m\le\frac{1}{2}\frac{\sqrt{m}}{(\log m)^2}\right\}$, and we have that

$$\left\{\Delta_m>\frac{\sqrt{m}}{(\log m)^2}\right\}\subset\bigcup_{j=1}^{m}\{X_j\ne X'_{mj}\},\quad\text{for } m\in T.$$

Thus we can get that

$$\sum_{m\ge1}\frac{d(m)(\log m)^b}{m}q_m$$
$$=\sum_{m\in T}\frac{d(m)(\log m)^b}{m}q_m+\sum_{m\notin T}\frac{d(m)(\log m)^b}{m}q_m=:I_1+I_2.$$

By taking the same way as above and Lemma 1, we can get the following two results.

For I_1,

$$\begin{aligned}I_1&\le\sum_{m\in T}\frac{d(m)(\log m)^b}{m}\sum_{j=1}^{m}P(X_j\ne X'_{mj})\\&\le\sum_{m\in T}d(m)(\log m)^bP(X_j\ne X'_{mj})\\&\le C_1+\frac{1+\delta}{(d-1)!}\sum_{m\in T\&m\ge k_0}(\log m)^{b+d-1}P\left(\|X\|>\frac{\sqrt{m}}{(\log m)^2}\right)\\&\le C_1+\frac{1+\delta}{(d-1)!}\sum_{m\ge1}(\log m)^{b+d-1}P\left(\|X\|>\frac{\sqrt{m}}{(\log m)^2}\right)\\&\le C_1+\frac{1+\delta}{(d-1)!}\sum_{m\ge1}(\log m)^{b+d-1}\\&\qquad\cdot\sum_{j\ge m}P\left(\frac{\sqrt{j}}{(\log j)^2}<\|X\|\le\frac{\sqrt{j+1}}{(\log(j+1))^2}\right)\\&\le C_1+\frac{1+\delta}{(d-1)!}\sum_{j\ge1}P\left(\frac{\sqrt{j}}{(\log j)^2}<\|X\|\le\frac{\sqrt{j+1}}{(\log(j+1))^2}\right)\\&\qquad\cdot\sum_{m=1}^{j}(\log m)^{b+d-1}\\&\le C_1+\frac{1+\delta}{(d-1)!}\sum_{j\ge1}P\left(\frac{\sqrt{j}}{(\log j)^2}<\|X\|\le\frac{\sqrt{j+1}}{(\log(j+1))^2}\right)\\&\qquad\cdot j(\log j)^{b+d-1}\\&\le C_1+CE[\|X\|^2(\log\|X\|)^{b+d+3}]<\infty.\end{aligned}$$

For I_2,

$$\begin{aligned}I_2&\le2\sum_{m\notin T}\frac{d(m)(\log m)^b}{m}T_m\frac{(\log m)^2}{\sqrt{m}}\\&\le C_2+\frac{2(1+\delta)}{(d-1)!}\sum_{m\notin T\&m\ge k_0}\left\{(\log m)^{b+d-1}m^{-1/2}\right.\\&\qquad\left.\cdot E\left[\|X\|I\left(\|X\|>\frac{\sqrt{m}}{(\log m)^2}\right)\right]\right\}\\&\le C_2+\frac{2(1+\delta)}{(d-1)!}\sum_{m\ge1}(\log m)^{b+d-1}m^{-1/2}\\&\qquad\cdot\sum_{j\ge m}E\left[\|X\|I\left(\frac{\sqrt{j}}{(\log j)^2}<\|X\|\le\frac{\sqrt{j+1}}{(\log(j+1))^2}\right)\right]\\&\le C_2+\frac{2(1+\delta)}{(d-1)!}\sum_{j\ge1}E\left[\|X\|I\left(\frac{\sqrt{j}}{(\log j)^2}<\|X\|\le\right.\right.\\&\qquad\left.\left.\frac{\sqrt{j+1}}{(\log(j+1))^2}\right)\right]\sum_{m=1}^{j}(\log m)^{b+d+1}m^{-1/2}\\&\le C_2+\frac{2(1+\delta)}{(d-1)!}\sum_{j\ge1}E\left[\|X\|I\left(\frac{\sqrt{j}}{(\log j)^2}<\|X\|\le\right.\right.\\&\qquad\left.\left.\frac{\sqrt{j+1}}{(\log(j+1))^2}\right)\right](\log j)^{b+d+1}j^{1/2}\\&\le C_2+CE[\|X\|^2(\log\|X\|)^{b+d+3}]<\infty.\end{aligned}$$

Thus we have proved the lemma.

Now we turn to the proof of Theorem 1.

Proof Let $\{Y'_{mj}\}$ be a sequence of independent H-valued Gaussian mean zero random variables with $\sum_m=Cov(Y'_{mj})=Cov(\bar{X}'_{mj})$, $1\le j\le m$. Set $T'_m=\sum_{j=1}^{m}Y'_{mj}$, $m\ge1$ and $\sum=Cov(X)$. Since $\sum-\sum_m$ is positive semidefinitive, thus by applying Corollary 3 of (Anderson, 1955), we have that for any $x\in\mathbb{R}$ and $m\in\mathbb{N}$,

$$P(\|T_m'\|\le x)\ge P(\|Y\|\le x/\sqrt{m}), \tag{6}$$

which together with Eq.(4) yields that

$$\begin{aligned}
&P\left(\|S_m\|\ge\varepsilon\sqrt{m\log m}\right)\\
&=P\left(\|S_m\|\ge\varepsilon\sqrt{m\log m}, \varDelta_m\le\frac{\sqrt{m}}{(\log m)^2}\right)\\
&\quad+P\left(\|S_m\|\ge\varepsilon\sqrt{m\log m}, \varDelta_m>\frac{\sqrt{m}}{(\log m)^2}\right)\\
&\le P\left(\|S_m\|\ge\varepsilon\sqrt{m\log m}, \varDelta_m\le\frac{\sqrt{m}}{(\log m)^2}\right)+q_m\\
&\le P\left(\|\bar{S}'_{mm}\|\ge\varepsilon\sqrt{m\log m}-\frac{\sqrt{m}}{(\log m)^2}\right)+q_m\\
&\le P\left(\|T'_m\|\ge\varepsilon\sqrt{m\log m}-\frac{2\sqrt{m}}{(\log m)^2}\right)+Cp_m+q_m \quad \text{by Eq.(4)}\\
&\le P\left(\|Y\|\ge(\varepsilon-2(\log m)^{-5/2})\sqrt{\log m}\right)+Cp_m+q_m \\
&\hspace{20em}\text{by Eq.(6)}\\
&\le P(\|Y\|\ge\varepsilon_1\sqrt{\log m})+Cp_m+q_m,
\end{aligned}$$

where $\varepsilon_1=\varepsilon-2(\log m)^{-5/2}\sim\varepsilon$ as $m\nearrow\infty$.

Thus, we have the upper bound below, via Proposition 1, Lemma 3 and Lemma 4,

$$\begin{aligned}
&\limsup_{\varepsilon\searrow 0}\varepsilon^{2(b+d)}\sum_N\frac{(\log|N|)^b}{|N|}P\left(\|S_N\|\ge\sigma\varepsilon\sqrt{|N|\log|N|}\right)\\
&=\limsup_{\varepsilon\searrow 0}\varepsilon^{2(b+d)}\sum_{m\ge1}\frac{d(m)(\log m)^b}{m}\cdot P\left(\|S_m\|\ge\sigma\varepsilon\sqrt{m\log m}\right)\\
&\le\frac{1}{(d-1)!}\frac{\sigma^{-2(b+d)}}{b+d}E[\|Y\|^{2(b+d)}].
\end{aligned}$$

Now we consider the lower bound of Eq.(3). First we consider the finite dimensional case, i.e. $l'<\infty$. Without loss of generality, we assume that the dimension of the space is l', for otherwise we can consider the projected space corresponding to the positive eigenvalues σ_i, i=1, 2, …, l'. Note that $\sum^{-1}$ exists since its eigenvalues are positive, and also $\sum_m\to\sum$, as $m\nearrow\infty$. Thus, we assume that $\sum_m^{-1}$ exists for all $m\ge1$.

Also define that

$$\varUpsilon_m=\|(\Sigma_m^{1/2}\Sigma^{-1/2})^{-1}\|=\sup_{y\ne0}\frac{\|(\Sigma_m^{1/2}\Sigma^{-1/2})^{-1}y\|}{\|y\|}. \tag{7}$$

Then, we have that

$$\begin{aligned}
&\|Y\|=\|(\Sigma_m^{1/2}\Sigma^{-1/2})^{-1}(\Sigma_m^{1/2}\Sigma^{-1/2})Y\|\\
&\le\|(\Sigma_m^{1/2}\Sigma^{-1/2})^{-1}\|\cdot\|\Sigma_m^{1/2}\Sigma^{-1/2}Y\|=\varUpsilon_m\|\Sigma_m^{1/2}\Sigma^{-1/2}Y\|,
\end{aligned}$$

and for any $x>0$,

$$\begin{aligned}
P(\|T'_m\|\ge x\sqrt{m})&=P(\|Y'_{mj}\|\ge x)\\
&=P(\|\Sigma_m^{1/2}\Sigma^{-1/2}Y\|\ge x)\ge P(\|Y\|\ge x\varUpsilon_m).
\end{aligned} \tag{8}$$

Noting Eq.(7), we also have that $\varUpsilon_m\to1$, as $m\nearrow\infty$. Thus, by Eq.(5) we can get

$$\begin{aligned}
&P\left(\|S_m\|\ge\varepsilon\sqrt{m\log m}\right)\\
&\ge P\left(\|S_m\|\ge\varepsilon\sqrt{m\log m}, \varDelta_m\le\frac{\sqrt{m}}{(\log m)^2}\right)\\
&\ge P\left(\|\bar{S}'_{mm}\|\ge\varepsilon\sqrt{m\log m}+\frac{\sqrt{m}}{(\log m)^2}, \varDelta_m\le\frac{\sqrt{m}}{(\log m)^2}\right)\\
&\ge P\left(\|\bar{S}'_{mm}\|\ge\varepsilon\sqrt{m\log m}+\frac{\sqrt{m}}{(\log m)^2}\right)-q_m\\
&\ge P\left(\|T'_m\|\ge\varepsilon\sqrt{m\log m}+\frac{2\sqrt{m}}{(\log m)^2}\right)-Cp_m-q_m\\
&\hspace{20em}\text{by Eq.(5)}\\
&\ge P\left[\|Y\|\ge\varUpsilon_m(\varepsilon+2(\log m)^{-5/2})\sqrt{\log m}\right]-Cp_m-q_m\\
&\hspace{20em}\text{by Eq.(8)}\\
&\ge P\left(\|Y\|\ge\varepsilon_2\sqrt{\log m}\right)-Cp_m-q_m,
\end{aligned}$$

where $\varepsilon_2=\varUpsilon_m(\varepsilon+2(\log m)^{-5/2})\sim\varepsilon$ as $m\nearrow\infty$. Hence, we have the following by Proposition 1, Lemma 3 and Lemma 4,

$$\begin{aligned}
&\liminf_{\varepsilon\searrow 0}\varepsilon^{2(b+d)}\sum_N\frac{(\log|N|)^b}{|N|}P\left(\|S_N\|\ge\sigma\varepsilon\sqrt{|N|\log|N|}\right)\\
&=\liminf_{\varepsilon\searrow 0}\varepsilon^{2(b+d)}\sum_{m\ge1}\frac{d(m)(\log m)^b}{m}\cdot P\left(\|S_m\|\ge\sigma\varepsilon\sqrt{m\log m}\right)\\
&\ge\frac{1}{(d-1)!}\frac{\sigma^{-2(b+d)}}{b+d}E[\|Y\|^{2(b+d)}].
\end{aligned}$$

Now, we consider the infinite dimensional case. Assume $l'=\infty$. For any $l''\geq l$, let Q: $H\to H$ be the mapping onto the l''-dimensional eigenspace of σ_i^2, $i=1, \ldots, l''$, i.e. $Q(y)=\sum_{i=1}^{l''}(y,e_i)e_i$, $y\in H$. Since $\|Q(y)\|\leq\|y\|$, $y\in H$, from the special case above, it follows that

$$\liminf_{\varepsilon\searrow 0}\varepsilon^{2(b+d)}\sum_{m\geq 1}\frac{d(m)(\log m)^b}{m}\cdot P\left(\|S_m\|\geq\sigma\varepsilon\sqrt{m\log m}\right)$$
$$\geq\liminf_{\varepsilon\searrow 0}\varepsilon^{2(b+d)}\sum_{m\geq 1}\frac{d(m)(\log m)^b}{m}\cdot P\left(\|Q(S_m)\|\geq\sigma\varepsilon\sqrt{m\log m}\right)$$
$$\geq\frac{1}{(d-1)!}\frac{\sigma^{-2(b+d)}}{b+d}E[\|Y\|^{2(b+d)}].$$

By letting $l''\to\infty$, we complete the proof of Theorem 1.

PROOF OF THEOREM 2

In this section, we set $B(\varepsilon)=\exp[1/(M\varepsilon^2)]$.

Lemma 5 For any $M\geq 4$, we have

$$\limsup_{\varepsilon\searrow 0}\varepsilon^{2(b+1)}\sum_{n\leq B(\varepsilon)}\frac{(\log n)^b}{n}\leq\frac{1}{b+1}M^{-(b+1)}.$$

Proof Obviously,

$$\limsup_{\varepsilon\searrow 0}\varepsilon^{2(b+1)}\sum_{n\leq B(\varepsilon)}\frac{(\log n)^b}{n}\leq\limsup_{\varepsilon\searrow 0}\varepsilon^{2(b+1)}\int_e^{B(\varepsilon)}\frac{(\log x)^b}{x}\mathrm{d}x$$
$$\leq\limsup_{\varepsilon\searrow 0}\varepsilon^{2(b+1)}\frac{1}{b+1}[\log B(\varepsilon)]^{b+1}=\frac{1}{b+1}M^{-(b+1)}.$$

Lemma 6 Let $\{X_n;\ n\geq 1\}$ be a sequence of i.i.d. H-valued random variables with $EX=0$ and $E\|X\|^2<\infty$. Suppose that f is a real function such that $|f(x)|\leq A$ and $|f'(x)|\leq A$. Then there exists a constant $C>0$ such that for any $b<0$, $0<\varepsilon<1/4$ and $\exp(1/\varepsilon^2)\leq m\leq l$, we have

$$Var\left\{\sum_{n=m}^{l}\frac{(\log n)^b}{n}f\left(\frac{\|S_n\|}{\varepsilon\sqrt{n\log n}}\right)\right\}$$
$$\leq CA^2\frac{(\log m)^{2b}}{m}+C\frac{A^2}{\varepsilon^2}(\log m)^{2b}.$$

Proof For $\exp(1/\varepsilon^2)\leq i<j$, we have that

$$Cov\left\{f\left(\frac{\|S_i\|}{\varepsilon\sqrt{i\log i}}\right),f\left(\frac{\|S_j\|}{\varepsilon\sqrt{j\log j}}\right)\right\}$$
$$=Cov\left\{f\left(\frac{\|S_i\|}{\varepsilon\sqrt{i\log i}}\right),f\left(\frac{\|S_{j+i}-S_i\|}{\varepsilon\sqrt{j\log j}}\right)\right\}$$
$$+Cov\left\{f\left(\frac{\|S_i\|}{\varepsilon\sqrt{i\log i}}\right),f\left(\frac{\|S_j\|}{\varepsilon\sqrt{j\log j}}\right)-f\left(\frac{\|S_{j+i}-S_i\|}{\varepsilon\sqrt{j\log j}}\right)\right\}$$
$$\leq\sqrt{Var\left\{f\left(\frac{\|S_i\|}{\varepsilon\sqrt{i\log i}}\right)\right\}E\left|f\left(\frac{\|S_j\|}{\varepsilon\sqrt{j\log j}}\right)-f\left(\frac{\|S_{j+i}-S_i\|}{\varepsilon\sqrt{j\log j}}\right)\right|^2}$$
$$\leq 2A^2\sqrt{\frac{E\|S_i\|^2}{\varepsilon^2 i\log i}\frac{E\|S_{j+i}-S_j-S_i\|^2}{\varepsilon^2 j\log j}}$$
$$\leq 2A^2\sqrt{\frac{E\|S_i\|^2}{\varepsilon^2 i\log i}\frac{E\|S_{j+i}-S_j\|^2+E\|S_i\|^2}{\varepsilon^2 j\log j}}$$
$$\leq CA^2\frac{\sqrt{i}}{\varepsilon^2\sqrt{\log i}\sqrt{j\log j}}.$$

Note that $\sum_{j=i+1}^{l}\frac{(\log j)^b}{j}\frac{1}{\sqrt{j\log j}}$ is of the order $(\log i)^{(2b-1)/2}/\sqrt{i}$. Hence, for $b<0$, we obtain that

$$Var\left\{\sum_{n=m}^{l}\frac{(\log n)^b}{n}f\left(\frac{\|S_n\|}{\varepsilon\sqrt{n\log n}}\right)\right\}$$
$$=\sum_{n=m}^{l}\frac{(\log n)^{2b}}{n^2}Var\left\{f\left(\frac{\|S_n\|}{\varepsilon\sqrt{n\log n}}\right)\right\}$$
$$+2\sum_{i=m}^{l}\sum_{j=i+1}^{l}\left\{\frac{(\log i)^b}{i}\frac{(\log j)^b}{j}\cdot Cov\left\{f\left(\frac{\|S_i\|}{\varepsilon\sqrt{i\log i}}\right),f\left(\frac{\|S_j\|}{\varepsilon\sqrt{j\log j}}\right)\right\}\right\}$$
$$\leq C\frac{A^2}{\varepsilon^2}\sum_{i=m}^{l}\sum_{j=i+1}^{l}\frac{(\log i)^b}{i}\frac{(\log j)^b}{j}\frac{\sqrt{i}}{\sqrt{\log i}\sqrt{j\log j}}$$
$$+CA^2\frac{(\log m)^{2b}}{m}$$
$$\leq CA^2\frac{(\log m)^{2b}}{m}+C\frac{A^2}{\varepsilon^2}\sum_{i=m}^{l}\frac{(\log i)^{2b-1}}{i}$$
$$\leq CA^2\frac{(\log m)^{2b}}{m}+C\frac{A^2}{\varepsilon}(\log m)^{2b}.$$

Now we turn to the proof of Theorem 2.

Proof Fix $M>4$ and $0<\eta<1/2$. Let f be a real function such that $I\{|x|\geq 1\}\leq f(x)\leq I\{|x|\geq 1-\eta\}$ and $\sup_x|f'(x)|<\infty$. Define $\varepsilon_k=1/k$, $k\geq 4M$. From Lemma 6, it follows that

$$Var\left\{\varepsilon_{k-1}^{2(b+1)}\sum_{n>B(\varepsilon_{k-1})}\frac{(\log n)^b}{n}f\left(\frac{\|S_n\|}{\varepsilon_k\sqrt{n\log n}}\right)\right\}$$
$$\leq C\varepsilon_{k-1}^{4(b+1)}\frac{[\log B(\varepsilon_{k-1})]^{2b}}{B(\varepsilon_{k-1})}+C\varepsilon_{k-1}^{4(b+1)}\varepsilon_k^{-2}[\log B(\varepsilon_{k-1})]^{2b}$$
$$\leq C\varepsilon_{k-1}^{4b+2}[\log B(\varepsilon_{k-1})]^{2b}\leq C\varepsilon_{k-1}^2\leq C/k^2.$$

Hence, for $-1<b<0$, by the Borel-Cantelli Lemma, we have that

$$\varepsilon_{k-1}^{2(b+1)}\sum_{n>B(\varepsilon_{k-1})}\left\{\frac{(\log n)^b}{n}\left[f\left(\frac{\|S_n\|}{\varepsilon_k\sqrt{n\log n}}\right)-Ef\left(\frac{\|S_n\|}{\varepsilon_k\sqrt{n\log n}}\right)\right]\right\}\to 0\quad L_2 \text{ and } a.s. \text{ as } k\to\infty.$$

Then, from Theorem 1 (by taking $d=1$), we get that

$$\limsup_{\varepsilon\searrow 0}\varepsilon^{2(b+1)}\sum_{n>B(\varepsilon)}\frac{(\log n)^b}{n}I\left(\|S_n\|\geq\varepsilon\sqrt{n\log n}\right)$$
$$\leq\limsup_{k\to\infty}\varepsilon_{k-1}^{2(b+1)}\sum_{n>B(\varepsilon_{k-1})}\frac{(\log n)^b}{n}I\left(\|S_n\|\geq\varepsilon_k\sqrt{n\log n}\right)$$
$$\leq\limsup_{k\to\infty}\varepsilon_k^{2(b+1)}\sum_{n>B(\varepsilon_{k-1})}\frac{(\log n)^b}{n}f\left(\frac{\|S_n\|}{\varepsilon_k\sqrt{n\log n}}\right)$$
$$\leq\limsup_{k\to\infty}\varepsilon_k^{2(b+1)}\sum_{n>B(\varepsilon_{k-1})}\frac{(\log n)^b}{n}Ef\left(\frac{\|S_n\|}{\varepsilon_k\sqrt{n\log n}}\right)\quad L_2 \text{ and } a.s.$$
$$\leq\limsup_{k\to\infty}\varepsilon_k^{2(b+1)}\sum_{n>B(\varepsilon_{k-1})}\left[\frac{(\log n)^b}{n}\cdot P\left(\|S_n\|\geq(1-\eta)\varepsilon_k\sqrt{n\log n}\right)\right]$$
$$\leq\limsup_{k\to\infty}\varepsilon_k^{2(b+1)}\sum_{n=1}^{\infty}\left[\frac{(\log n)^b}{n}P\left(\|S_n\|\geq(1-\eta)\varepsilon_k\sqrt{n\log n}\right)\right]$$
$$\leq(1-\eta)^{-2(b+1)}\frac{1}{b+1}E[\|Y\|]^{2(b+1)}.$$

Also by Lemma 5 and the arbitrariness of η, we have

$$\limsup_{\varepsilon\searrow 0}\varepsilon^{2(b+1)}\sum_{n=1}^{\infty}\frac{(\log n)^b}{n}I\left(\|S_n\|\geq\varepsilon\sqrt{n\log n}\right)\leq\frac{1}{b+1}E[\|Y\|]^{2(b+1)}\quad L_2 \text{ and } a.s.$$

Similarly, we can easily get that

$$\liminf_{\varepsilon\searrow 0}\varepsilon^{2(b+1)}\sum_{n=1}^{\infty}\frac{(\log n)^b}{n}I\left(\|S_n\|\geq\varepsilon\sqrt{n\log n}\right)\geq\frac{1}{b+1}E[\|Y\|]^{2(b+1)}\quad L_2 \text{ and } a.s.$$

Thus the proof is complete.

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