Journal of Zhejiang University SCIENCE A
ISSN 1673-565X (Print); ISSN 1862-1775 (Online)
www.zju.edu.cn/jzus; www.springerlink.com
E-mail: jzus@zju.edu.cn

Construction of solitonary and periodic solutions to some nonlinear equations using EXP-function method

ZHANG Mei[†], ZHANG Wen-jing
(*Department of Applied Physics, College of Science, Donghua University, Shanghai 201620, China*)
[†]E-mail: zhmm@mail.dhu.edu.cn
Received Aug. 16, 2006; revision accepted Dec. 17, 2006

Abstract: This paper applies the EXP-function method to find exact solutions of various nonlinear equations. Tzitzeica-Dodd-Bullough (TDB) equation was selected to illustrate the effectiveness and convenience of the suggested method. More generalized solitonary solutions with free parameters were obtained by suitable choice of the free parameters, and also the obtained solitonary solutions can be converted into periodic solutions.

Key words: Soliton, Periodic solution, Nonlinear equation, EXP-function method
doi:10.1631/jzus.2007.A0660 **Document code:** A **CLC number:** O175.29; O241.7

INTRODUCTION

Recently exact solutions including solitary solutions, periodic solutions, and compacton-like solutions for nonlinear equations have attracted much attention. Many new approaches with merits on the one hand and disadvantages on the other hand were introduced to solve various nonlinear equations; for examples, tanh-function method (Fan and Hon, 2003; Wazwaz, 2005; Abdusalam, 2005; El-Sabbagh and Ali, 2005; Bai and Zhao, 2006), F-method (Zhang *et al*., 2006; Wang and Li, 2005; Yomba, 2005a; 2005b; 2005c; Ren and Zhang, 2006; Wang and Zhang, 2005), variational iteration method (Wang *et al*., 2005; He and Wu, 2006a; 2006b; Odibat and Momani, 2006; Bildik and Konuralp, 2006; Momani and Abuasad, 2006), homotopy perturbation method (He, 2005a; 2005b; 2006a; 2006b; 2006c; El-Shahed, 2005; Siddiqui *et al*., 2006), and variational approach (Liu, 2004; 2005; Momani and Abuasad, 2006; He and Wu, 2006a; 2006b). Complete reviews are given in (Rosenau and Hyman, 1993; Rosenau, 2000; 2006; Pikovsky and Rosenau, 2006).

In this paper we apply a novel method called EXP-function method to search for exact solutions of various nonlinear equations.

PRELIMINARIES

Considering a general nonlinear partial differential equation in the form

$$F(u,u_x,u_y,u_z,u_t,u_{xx},u_{yy},u_{zz},u_{tt},u_{xy},u_{xt},u_{yt},...)=0. \tag{1}$$

Using a transformation

$$\eta = ax+by+cz+dt, \tag{2}$$

where a, b, c and d are constants, we can rewrite Eq.(1) in the following nonlinear ordinary differential equation:

$$G(u,u',u'',u''',...)=0, \tag{3}$$

where the prime denotes the derivation with respect to η.

The EXP-function method is very simple and straightforward, and is based on a priori assumption that travelling wave solutions can be in the form

$$u(\eta)=\sum_{n=-c}^{d} a_n \exp(n\eta) \Bigg/ \sum_{m=-f}^{g} a_m \exp(m\eta), \tag{4}$$

where c, d, f and g are positive integers which could be freely chosen, a_n and b_m are unknown constants to be determined.

RESULT ANALYSIS

Wazwaz (2005) applied the tanh method to some special equations which are very difficult to solve. One Wazwaz's example is used here to illustrate the effectiveness and convenience of the suggested method; comparison with Wazwaz's results will also be made.

The Tzitzeica-Dodd-Bullough (TDB) equation

The TDB equation is of the form (Brezhnev, 1996; Wazwaz, 2005)

$$u_{xt} - e^{-u} - e^{-2u} = 0. \tag{5}$$

Uses the transformation

$$v = e^{-u}, \tag{6}$$

so $u=\ln(1/v)$, and Eq.(5) becomes

$$-vv_{xt} + v_x v_t - v^3 - v^4 = 0. \tag{7}$$

Introducing a complex variable defined as

$$\eta = kx + \omega t, \tag{8}$$

we have

$$-k\omega vv'' + k\omega(v')^2 + v^3 - v^4 = 0, \tag{9}$$

where prime denotes differentiation with respect to η.

According to the EXP-function method (He and Wu, 2006b; He and Abdou, 2006), we assume that the solution of Eq.(9) can be expressed in the form

$$v(\eta) = \frac{a_1 \exp\eta + a_0 + a_{-1}\exp(-\eta)}{\exp\eta + b_0 + b_{-1}\exp(-\eta)}. \tag{10}$$

Substituting Eq.(10) into Eq.(9), by MATLAB we obtain

$$\frac{1}{A}\sum_{i=-4}^{4} C_i \exp(i\eta) = 0, \tag{11}$$

where

$$\begin{aligned}
&A = [\exp\eta + b_{-1}\exp(-\eta) + b_0]^4,\\
&C_4 = -a_1^4 - a_1^3,\\
&C_3 = -3a_1^2 a_0 - a_1^3 B_0 - 4a_1^3 a_0 + k\omega a_1^2 b_0 - k\omega a_0 a_1,\\
&C_2 = -6a_1^2 a_0^2 + 4k\omega a_1^2 b_{-1} - 3a_1^2 a_0 b_0 - a_1^3 b_{-1}\\
&\qquad - 3a_1 a_0^2 - 3a_1^2 a_{-1} - 4a_1^3 a_{-1} - 4k\omega a_1 a_{-1},\\
&C_1 = -6k\omega a_1 b_0 a_{-1} - k\omega a_0 a_1 b_0^2 - 12a_1^2 a_{-1} a_0 - 6a_1 a_{-1} a_0\\
&\qquad - 4a_1 a_0^3 - 3a_1 a_0^2 b_0 - a_0^3 + 6k\omega a_1 b_{-1} a_0 + k\omega a_1^2 b_{-1} b_0\\
&\qquad + k\omega a_0^2 b_0 - 3a_1^2 a_{-1} b_0 - k\omega a_{-1} a_0 - 3a_1^2 a_0 b_{-1},\\
&C_0 = -a_0^4 - 3a_1 a_0^2 a_{-1} - 3a_1^2 a_{-1} b_{-1} - a_0^3 b_0 + 4k\omega a_0^2 b_{-1}\\
&\qquad - 3a_{-1} a_0^2 - 3a_1 a_{-1}^2 - 6a_1 a_{-1} a_0 b_0 - 6a_1^2 a_{-1}^2\\
&\qquad - 12a_1 a_{-1} a_0^2 - 4k\omega a_1 a_{-1} b_0^2,\\
&C_{-1} = k\omega a_{-1}^2 b_0 - 12a_1 a_{-1}^2 a_0 - k\omega a_{-1} a_0 b_0^2\\
&\qquad - 6k\omega a_1 b_{-1} a_{-1} b_0 - a_0^2 b_{-1} - 3a_1 a_{-1}^2 b_0\\
&\qquad - 3a_{-1}^2 a_0 - 3a_{-1} a_0^2 b_0 - 4a_{-1} a_0^3 - 6a_1 a_{-1} a_0 b_{-1}\\
&\qquad - k\omega a_1 b_{-1}^2 a_0 + 6k\omega a_{-1} a_0 b_{-1} + k\omega a_0^2 b_{-1} b_0,\\
&C_{-2} = 4k\omega a_{-1}^2 b_{-1} - 3a_1 a_{-1}^2 b_{-1} - 4k\omega a_1 a_{-1} b_{-1}^2\\
&\qquad - 4a_1 a_{-1}^3 - 6a_{-1}^2 a_0^2 - 3a_{-1} a_0^2 b_{-1} - 3a_{-1}^2 a_0 b_0 - a_{-1}^3,\\
&C_{-3} = -3a_{-1}^2 a_0 b_{-1} - k\omega a_{-1} a_0 b_{-1}^2 + k\omega a_{-1}^2 b_{-1} b_0\\
&\qquad - 4a_{-1}^3 a_0 - a_{-1}^3 b_0,\\
&C_{-4} = -a_{-1}^3 b_{-1} - a_{-1}^4.
\end{aligned}$$

Equating the coefficients of $\exp(n\eta)$ to zero, we have

$$C_i = 0, \quad -4 \le i \le 4, \ i \in \mathbb{R}. \tag{12}$$

Solving the system Eq.(12), we obtain the following three sets of solutions:

$$\begin{cases}
a_1 = 0,\ a_{-1} = -b_{-1},\ a_0 = 0,\ b_0 = 0,\ \omega = -1/(4k) \\
\qquad (b_{-1} \text{ and } k \text{ are free parameters}), & (13.1)\\
a_1 = 0,\ a_{-1} = 4a_0^2,\ b_0 = 3a_0,\ b_{-1} = -4a_0^2,\ \omega = -1/k \\
\qquad (b_{-1} \text{ and } k \text{ are free parameters}), & (13.2)\\
a_1 = 0,\ a_{-1} = 0,\ a_0 = k\omega b_0,\ b_{-1} = b_0^2(1 + k\omega)/4 \\
\qquad (b_0, k \text{ and } \omega \text{ are free parameters}). & (13.3)
\end{cases}$$

Solution of TDB equation

From Eq.(13.1), we obtain the following exact solution

$$v(x,t) = \frac{-b_{-1}\exp[-kx + t/(4k)]}{\exp[kx - t/(4k)] + b_{-1}\exp[-kx - t/(4k)]}, \tag{14}$$

where k and b_{-1} are non-zero free parameters.

Applying tanh-function method, Wazwaz obtained the following solution

$$v(x,t)=-\left\{1\pm\tanh\left[(x-ct)\big/(2\sqrt{c})\right]\right\}\big/2$$
$$=\frac{-\exp\left[\pm(x-ct)\big/(2\sqrt{c})\right]}{\exp\left[x\big/(2\sqrt{c})-\sqrt{c}t\big/2\right]+\exp\left[-x\big/(2\sqrt{c})+\sqrt{c}t\big/2\right]}. \tag{15}$$

In case $b_{-1}=1$, our solution Eq.(14) turns out to be Wazwaz's. The solution of the TDB equation therefore can be expressed as follows:

$$u(x,t)=-\ln v(x,t)$$
$$=\ln\frac{\exp[kx-t/(4k)]+b_{-1}\exp[kx-t/(4k)]}{-b_{-1}\exp[-kx+t/(4k)]} \tag{16}$$
$$=\ln\{-1-\exp[2kx-t/(2k)]/b_{-1}\}.$$

Here k and b_{-1} are non-zero free parameters and requires that $b_{-1}<0$ and $2kx-t/(2k)>\ln(-b_{-1})$.

Eq.(13.2) corresponds to the following exact solution

$$v(x,t)=\frac{4a_0^2\exp(-kx+t/k)+a_0}{\exp(kx-t/k)-4a_0^2\exp(-kx+t/k)+3a_0}, \tag{17}$$

where k and a_0 are free parameters. So

$$u(x,t)=-\ln v(x,t)$$
$$=\ln\frac{\exp(kx-t/k)-4a_0^2\exp(-kx+t/k)+3a_0}{4a_0^2\exp(-kx+t/k)+a_0}$$
$$=\ln\left[-1+\frac{\exp(kx-t/k)}{a_0}\times\frac{\exp(kx-t/k)+4a_0}{4a_0+\exp(kx-t/k)}\right] \tag{18}$$
$$=\ln\left[-1+\frac{\exp(kx-t/k)}{a_0}\right].$$

By Eq.(13.3), we obtain another solution, which reads

$$v(x,t)=\frac{k\omega b_0}{\exp(kx+\omega t)+b_0^2(1+k\omega)\exp(-kx-\omega t)\big/4+b_0}, \tag{19}$$

and

$$u(x,t)=\ln[v^{-1}(x,t)]$$
$$=\ln\left[\frac{\exp(kx+\omega t)+b_0^2(1+k\omega)\exp(-kx-\omega t)\big/4+b_0}{k\omega b_0}\right]. \tag{20}$$

Our solutions containing some free parameters are in generalized forms, and include special solutions obtained by other methods, for example, tanh-function method. If the free parameters are chosen to be imaginary numbers, the obtained solitonary solutions can be converted into periodic solutions.

Let $\omega=\mathrm{i}W$, $k=\mathrm{i}K$ in Eq.(20), and using the identity $\exp(\mathrm{i}Kx+\mathrm{i}Wt)=\cos(Kx+Wt)+\mathrm{i}\sin(Kx+Wt)$, we have

$$u(x,t)=\ln\left\{\{[1+b_0{}^2(1-KW)/4]\cos(Kx+Wt)+\right.$$
$$\left.\mathrm{i}[1-b_0{}^2(1-KW)/4]\sin(Kx+Wt)+b_0\}\big/(-KWb_0)\right\}. \tag{21}$$

Elimination of the imaginary part requires that

$$\frac{b_0{}^2}{4}(1-KW)=1,\ \text{ or }\ 1-\frac{4}{b_0{}^2}=KW. \tag{22}$$

We therefore obtain a periodic solution for the equation, which reads

$$u(x,t)=\ln\left\{\frac{2\cos[Kx-(4b_0^{-2}-1)t\big/K]+b_0}{b_0(4b_0^{-2}-1)}\right\}, \tag{23}$$

where K and b_0 are free real numbers. Fig.1 illustrates its periodic solutions with different values of b_0.

Instead of Eq.(10), we can also assume that the solution can be expressed in a more general form:

$$v(\eta)=\frac{a_1\mathrm{e}^{\eta}+a_0+a_{-1}\mathrm{e}^{-\eta}+a_{-2}\mathrm{e}^{-2\eta}}{b_1\mathrm{e}^{\eta}+b_0+b_{-1}\mathrm{e}^{-\eta}+b_{-2}\mathrm{e}^{-2\eta}}. \tag{24}$$

After identification of the parameters in Eq.(24) by the same manner as illustrated above, we obtain the following solution

$$v(x,t)=\left[(a_{-1}b_{-1}+a_{-1}^2+b_{-2}b_0)/(-b_{-2})+a_{-1}\mathrm{e}^{-\eta}-\right.$$
$$\left.b_{-2}\mathrm{e}^{-2\eta}\right]\Big/\left[(a_{-1}b_{-1}+a_{-1}^2+b_{-2}b_0)(b_{-1}+a_{-1})\cdot\right. \tag{25}$$
$$\left.(b_{-2})^{-2}\mathrm{e}^{\eta}+b_0+b_{-1}\mathrm{e}^{-\eta}+b_{-2}\mathrm{e}^{-2\eta}\right],$$

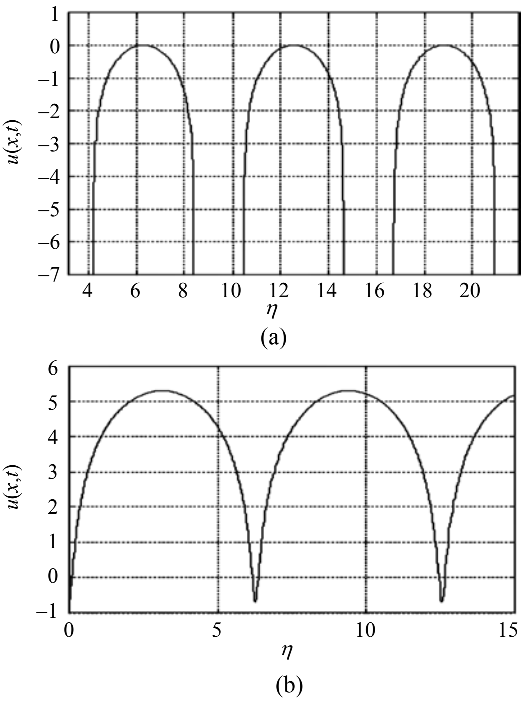

Fig.1 Periodic solutions of the TDB equation.
(a) b_0=1; (b) b_0=−2.01

where a_1, b_1, b_2 and k are free parameters, and $\eta=kx-t/k$. By suitable choice of the parameters, the solution Eq.(25) can be converted into the above obtained solutions.

If we choose the trial-function in the form

$$v(x,t)=(a_2\,\mathrm{e}^{2\eta}+a_1\,\mathrm{e}^{\eta}+a_0+a_{-1}\,\mathrm{e}^{-\eta}+a_{-2}\,\mathrm{e}^{-2\eta})\cdot (b_2\,\mathrm{e}^{2\eta}+b_0+b_{-2}\,\mathrm{e}^{-2\eta})^{-1}, \tag{26}$$

we can obtain the following solution by a similar operation:

$$v(x,t)=\frac{\dfrac{(1+\omega k)(\omega k)^2 b_0^2}{a_{-1}(-1+\omega k)^2}\mathrm{e}^{\eta}-\dfrac{2\omega k b_0}{-1+\omega k}+a_{-1}\,\mathrm{e}^{-\eta}}{\dfrac{(1+\omega k)^2(\omega k)^2 b_0^3}{2a_{-1}^2(-1+\omega k)^3}\mathrm{e}^{2\eta}+b_0+\dfrac{(-1+\omega k)a_{-1}^2}{2(\omega k)^2 b_0}\mathrm{e}^{-2\eta}}. \tag{27}$$

DISCUSSION AND CONCLUSION

We give a very simple and straightforward method called EXP-function method for nonlinear wave equations. We make some important remarks on the method as follows:

(1) The solutions do not strongly depend on the trial-function, so we can always choose the solution in the form

$$v(\eta)=\frac{a_1\,\mathrm{e}^{\eta}+a_0+a_{-1}\,\mathrm{e}^{-\eta}}{\mathrm{e}^{\eta}+b_0+b_{-1}\,\mathrm{e}^{-\eta}}.$$

(2) The method leads to both the generalized solitonary solutions and periodic solutions.

(3) The obtained solutions include naturally those obtained by tanh-function method.

(4) The solution procedure by the help of MATLAB or Mathematica is of utter simplicity.

(5) The method can be easily applied to many kinds of nonlinear equations, and this letter can be used as paradigm for real-life physics problems.

References

Abdusalam, H.A., 2005. On an improved complex tanh-function method. *Int. J. Nonlinear Sci. Num. Simulation*, **6**(2): 99-106.

Bai, C.L., Zhao, H., 2006. Generalized extended tanh-function method and its application. *Chaos, Solitons & Fractals*, **27**:1026-1035. [doi:10.1016/j.chaos.2005.04.069]

Bildik, N., Konuralp, A., 2006. The use of Variational Iteration Method, Differential Transform Method and Adomian Decomposition Method for solving different types of nonlinear partial differential equations. *Int. J. Nonlinear Sci. Num. Simulation*, **7**:65-70.

Brezhnev, Y.V., 1996. Darboux transformation and some multi-phase solutions of the Tzitzeica-Dodd-Bullough equation. *Phys. Lett. A*, **211**:94-100. [doi:10.1016/0375-9601(95)00955-8]

El-Shahed, M.F., 2005. Application of He's homotopy perturbation method to Volterra's integro-differential equation. *Int. J. Nonlinear Sci. Num. Simulation*, **6**(2):163-168.

El-Sabbagh, M.F., Ali, A.T., 2005. New exact solutions for (3+1)-dimensional Kadomtsev-Petviashvili equation and generalized (2+1)-dimensional Boussinesq equation. *Int. J. Nonlinear Sci. Num. Simulation*, **6**:151-162.

Fan, E., Hon, Y.C., 2003. Applications of extended tanh method to 'special' types of nonlinear equations. *Appl. Math. Comput.*, **141**:351-358. [doi:10.1016/S0096-3003(02)00260-6]

He, J.H., 2005a. Homotopy perturbation method for bifurcation of nonlinear problems. *Int. J. Nonlinear Sci. Num. Simulation*, **6**(2):207-208.

He, J.H., 2005b. Application of homotopy perturbation method to nonlinear wave equations. *Chaos, Solitons & Fractals*, **26**:695-700. [doi:10.1016/j.chaos.2005.03.006]

He, J.H., 2006a. New interpretation of homotopy perturbation method. *Int. J. Modern Phys. B*, **20**(18):2561-2568. [doi:10.1142/S0217979206034819]

He, J.H., 2006b. Homotopy perturbation method for solving boundary value problems. *Phys. Lett. A*, **350**:87-88. [doi:10.1016/j.physleta.2005.10.005]

He, J.H., 2006c. Some asymptotic methods for strongly nonlinear equations. *Int. J. Modern Phys. B*, **20**:1141-1199. [doi:10.1142/S0217979206033796]

He, J.H., Abdou, M.A., 2006. New periodic solutions for nonlinear evolution equations using Exp-function method. *Chaos, Solitons & Fractals*, in press. [doi:10.1016/j.chaos.2006.05.072]

He, J.H., Wu, X.H., 2006a. Construction of solitary solution and compacton-like solution by variational iteration method. *Chaos, Solitons & Fractals*, **29**:108-113. [doi:10.1016/j.chaos.2005.10.100]

He, J.H., Wu, X.H., 2006b. Exp-function method for nonlinear wave equations. *Chaos, Solitons & Fractals*, **30**:700-708. [doi:10.1016/j.chaos.2006.03.020]

Liu, H.M., 2004. Variational approach to nonlinear electrochemical system. *Int. J. Nonlinear Sci. Num. Simulation*, **5**(1):95-96.

Liu, H.M., 2005. Generalized variational principles for ion acoustic plasma waves by He's semi-inverse method. *Chaos, Solitons & Fractals*, **23**:573-576. [doi:10.1016/j.chaos.2004.05.005]

Momani, S., Abuasad, S., 2006. Application of He's variational iteration method to Helmholtz equation. *Chaos, Solitons & Fractals*, **27**:1119-1123. [doi:10.1016/j.chaos.2005.04.113]

Odibat, Z.M., Momani, S., 2006. Application of variational iteration method to nonlinear differential equations of fractional order. *Int. J. Nonlinear Sci. Num. Simulation*, **7**:27-34.

Pikovsky, A., Rosenau, P., 2006. Phase compactons. *Phys. D: Nonlinear Phenomena*, **218**(1):56-69. [doi:10.1016/j.physd.2006.04.015]

Ren, Y.J., Zhang, H.Q., 2006. A generalized F-expansion method to find abundant families of Jacobi Elliptic Function solutions of the (2+1)-dimensional Nizhnik-Novikov-Veselov equation. *Chaos, Solitons & Fractals*, **27**:959-979. [doi:10.1016/j.chaos.2005.04.063]

Rosenau, P., Hyman, J.M., 1993. Compactons: solitons with finite wavelengths. *Phys. Rev. Lett.*, **75**(5):564-567. [doi:10.1103/PhysRevLett.70.564]

Rosenau, P., 2000. Compact and noncompact dispersive patterns. *Phys. Lett. A*, **275**(3):193-203. [doi:10.1016/S0375-9601(00)00577-6]

Rosenau, P., 2006. On a model equation of traveling and stationary compactons. *Phys. Lett. A*, **356**:44-50. [doi:10.1016/j.physleta.2006.03.033]

Siddiqui, A.M., Mahmood, R., Ghori, Q.K., 2006. Thin film flow of a third grade fluid on a moving belt by He's homotopy perturbation method. *Int. J. Nonlinear Sci. Num. Simulation*, **7**:7-14.

Wang, D.S., Zhang, H.Q., 2005. Further improved F-expansion method and new exact solutions of Konopelchenko-Dubrovsky equation. *Chaos, Solitons & Fractals*, **25**:601-610. [doi:10.1016/j.chaos.2004.11.026]

Wang, M.L., Li, X.Z., 2005. Extended F-expansion method and periodic wave solutions for the generalized Zakharov equations. *Phys. Lett. A*, **343**:48-54. [doi:10.1016/j.physleta.2005.05.085]

Wang, Q., Chen, Y., Zhang, H., 2005. A new Riccati equation rational expansion method and its application to (2+1)-dimensional Burgers equation. *Chaos, Solitons & Fractals*, **25**(5):1019-1028. [doi:10.1016/j.chaos.2005.01.039]

Wazwaz, A.M., 2005. The tanh method: solitons and periodic solutions for the Dodd-Bullough-Mikhailov and the Tzitzeica-Dodd-Bullough equations. *Chaos, Solitons & Fractals*, **25**(1):55-63. [doi:10.1016/j.chaos.2004.09.122]

Yomba, E., 2005a. The extended Fan's sub-equation method and its application to KdV-MKdV, BKK and variant Boussinesq equations. *Phys. Lett. A*, **336**(6):463-476. [doi:10.1016/j.physleta.2005.01.027]

Yomba, E., 2005b. The extended F-expansion method and its application for solving the nonlinear wave, CKGZ,GDS, DS and GZ equations. *Phys. Lett. A*, **340**(1-4):149-160. [doi:10.1016/j.physleta.2005.03.066]

Yomba, E., 2005c. The modified extended Fan sub-equation method and its application to (2+1)-dimensional dispersive long wave equation. *Chaos, Solitons & Fractals*, **26**:785-794.

Zhang, J.L., Wang, M.L., Wang, Y.M., Fang, Z.D., 2006. The improved F-expansion method and its applications. *Phys. Lett. A*, **350**:103-109. [doi:10.1016/j.physleta.2005.10.099]