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Existence and uniqueness theorem for flow and heat transfer of a non-Newtonian fluid over a stretching sheet

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Abstract: Analysis is carried out to study the existence, uniqueness and behavior of exact solutions of the fourth order nonlinear coupled ordinary differential equations arising in the flow and heat transfer of a viscoelastic, electrically conducting fluid past a continuously stretching sheet. The ranges of the parametric values are obtained for which the system has a unique pair of solutions, a double pair of solutions and infinitely many solutions.

Key words: Viscoelastic fluid, Stretching sheet, MHD flow, Heat transfer, Nonlinear systems, Existence, Uniqueness
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INTRODUCTION

Steady laminar boundary layer flow over a stretching sheet has received considerable attention due to its many theoretical and technical applications in the engineering and technology field. Some of these applications include aerodynamic extrusion of plastic sheets, the boundary layer along a material handling conveyers, cooling of an infinite metallic plate in a cooling bath, the boundary layer along a liquid film in a condensation process and heat treated materials that travel between feed and wind-up rollers. In view of these applications, Sakiadis (1961) initiated the study of boundary layer flow over a continuous solid surface moving at constant speed and then extended to stretching sheet by McCormack and Crane (1973). Following them Gupta and Gupta (1977) examined the heat and mass transfer using a similarity transformation for the boundary layer flow over a stretching sheet subject to suction or blowing. Chen and Char (1988) investigated the effects of power law heat flux variation on the heat transfer characteristics of a continuous linearly stretching sheet subject to suction or blowing. However, the above researches are restricted to flows of Newtonian fluids.

Many materials such as polymer solutions or melts, drilling mud, clastomers, certain oils, greases and many other emulsions are classified as non-Newtonian fluids due to the nonlinearity in the relationship between the stress and the rate of strain of these fluids. There are many models describing the properties but not all of non-Newtonian fluids. These models or constitutive equations, however cannot describe all the behaviors of these non-Newtonian fluids, for example, stress differences, shear thinning or shear thickening, stress relaxation, elastic effects, memory effects, etc. Among these models the fluids of differential type, for example fluids of second grade, third grade and fourth grade have received much attention in the past due to their elegance and simplicity. Non-Newtonian fluids have been of interest in industries. Fox *et al.*(1969) studied the flow of a non-Newtonian fluid characterized by a power law model. Rajagopal *et al.*(1984) analyzed the effects of viscoelasticity on the flow of a second grade fluid over a stretching sheet whose constitutive equation is given by

$$T = -pI + \mu A_1 + \alpha_1 A_2 + \alpha_2 A_1^2, \quad (1)$$

where T is the Cauchy stress tensor, the spherical stress $-pI$ is due to the constraint of incompressibility, μ is the viscosity, α_1 and α_2 are the material modules usually referred to as normal stress modules and A_1 and A_2 are the first two Rivlin-Ericksen tensors defined by (Rivlin and Ericksen, 1955)

$$A_1 = (\nabla V) + (\nabla V)^{\mathrm{T}}, A_2 = \frac{\mathrm{d}A_1}{\mathrm{d}t} + A_1(\nabla V) + (\nabla V)^{\mathrm{T}} A_1. \quad (2)$$

If the fluid of second grade is to be compatible with thermodynamics in the sense that all motions of the fluid meet the Clausius-Duhem inequality and the assumption that the specific Helmholtz free energy of the fluid is a minimum, then (Dunn and Fosdick, 1974)

$$\mu \geq 0, \alpha_1 \geq 0, \alpha_1 + \alpha_2 = 0. \quad (3)$$

Fosdick and Rajagopal (1979) demonstrated that if $\alpha_1 \leq 0$ while the other two restrictions hold, the fluid exhibits unacceptable instability characteristics. We do not intend to discuss it, since a critical review of Dunn and Rajagopal (1995) has already given a concise discussion about the issue.

Important theoretical studies of second grade fluids were conducted by Hayat *et al.*(2004a; 2004b). Cortell (2006a; 2006b) and Hayat and Sajid (2007) studied the flow and heat transfer of a second grade fluid over stretching sheet. One can further refer to the works of eminent researchers (Hayat *et al.*, 2006; Khan and Sanjayanand, 2005; Liu, 2004; Vajravelu and Soewono, 1996; Vajravelu and Rollins, 2004) regarding the flow of non-Newtonian fluids over stretching sheet with diverse physical effects. In this work we study the existence, uniqueness and behavior of exact solutions of fourth order nonlinear coupled ordinary differential equations arising in the flow and heat transfer of an electrically conducting second grade fluid past a continuously stretching sheet.

FORMULATION OF THE PROBLEM

We consider the flow an incompressible electrically conducting fluid of second grade, obeying Eqs.(1)~(3) subjected to a transverse uniform magnetic field $B=(0,B_0,0)$, over a semi infinite stretching sheet coinciding with the plane $y=0$. The sheet is stretched (Fig.1) horizontally by pulling on both sides with equal forces parallel to the sheet keeping the origin fixed at speed u varying with the distance from the slit ($u=Cx$). Furthermore, we confine our attention to the flow in the region, $x\geq 0$, $y>0$. Following (Liu, 2004), the basic boundary layer equations for the steady flow and heat transfer with internal heat generation or absorption in the usual notation are,

$$(f')^2 - ff'' = f''' + \lambda(2f'f''' - (f'')^2 - ff'''') - Mf', \quad (4)$$
$$\theta'' + \sigma f\theta' - \sigma(f' - \alpha)\theta = 0. \quad (5)$$

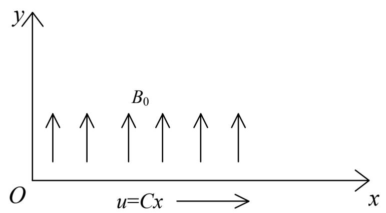

Fig.1 Schematic diagram of the flow domain

The relevant boundary conditions are,

$$f(0) = 0, f'(0) = 1, \theta(0) = 1,$$
$$f'(\eta) \to 0, f''(\eta) \to 0, \theta(\eta) \to 0,$$
$$\text{as } \eta \to \infty, \quad (6)$$

where $\eta=(C/\nu)^{1/2}y$ is the non-dimensional distance, $\nu=\mu/\rho$ is the kinematic viscosity, $\lambda=\lambda_1 C/\nu$ is the viscoelastic parameter, $M=\sigma_0 B_0^2/(\rho C)$ is the magnetic parameter, $\sigma=\mu C_p/k$ is the Prandtl number and $\alpha=Q/(C\rho C_p)$ is the heat source or sink parameter and a prime denotes differentiation with respect to η.

ANALYSIS OF EXISTENCE AND UNIQUENESS

We would now like to study the existence and uniqueness results for the system of Eqs.(4)~(5) with boundary conditions Eq.(6). Knowing that for a certain choice of parameters exponential type solutions exist, we generate series solutions with exponential terms. This idea was used by Kichenassamy and

Olver (1992). It can be shown by direct substitution that the system Eqs.(4)~(6) has solutions of the form,

$$f(\eta)=a_0+a_1\mathrm{e}^{-\beta\eta},\quad \theta(\eta)=b_1\mathrm{e}^{-\beta\eta},$$

where $\beta=\sqrt{(1+M)/(1-\lambda)}$ is a combined parameter relating the effects of viscoelasticity of the second grade fluid and the magnetic field. It turns out later in this section that there are many solutions of different forms exist for the system Eqs.(4)~(6). We look for the solutions of the form,

$$f(\eta)=\sum_{n=0}^{\infty}a_n\mathrm{e}^{-n\beta\eta},\ \theta(\eta)=\sum_{n=1}^{\infty}b_n\mathrm{e}^{-n\beta\eta}. \tag{7}$$

Substituting Eq.(7) into Eqs.(4) and (5) and equating the like terms in the exponentials $\mathrm{e}^{-n\beta\eta}$, we obtain,

$$\beta^2(-\beta^2\lambda a_0+\beta-a_0)a_1-M\beta a_1=0, \tag{8}$$

$$\beta^2n^2(-n^2\beta^2\lambda a_0+n\beta-a_0)a_n-M\beta a_n$$
$$=\sum_{k=1}^{n-1}[(n-k)(n-2k)\beta^2+(n-k)^2(n-2k)^2\lambda\beta^4]a_ka_{n-k},n\geq2, \tag{9}$$

$$(\beta^2-\sigma a_0\beta+\sigma\alpha)b_1=0, \tag{10}$$

$$(n^2\beta^2-n\sigma a_0\beta+\sigma\alpha)b_n=\sum_{k=1}^{n-1}\sigma\beta(n-2k)a_kb_{n-k},\ n\geq2. \tag{11}$$

Now the problem is reduced to solving Eqs.(8)~(11) along with the boundary conditions Eq.(6). If $a_1\neq0$, we get from Eq.(8),

$$\lambda a_0\beta^3-\beta^2+a_0\beta+M=0. \tag{12}$$

The right hand side of Eq.(9) becomes zero if n=2. This implies that, if in addition to Eq.(12),

$$\beta^2n^2(-n^2\beta^2\lambda a_0+n\beta-a_0)-M\beta\neq0,\ n\geq2. \tag{13}$$

We then obtain a_2=0 and therefore (from Eq.(9)) a_n=0, $n\geq2$. The solution to Eq.(4) satisfying the above conditions is of the form

$$f(\eta)=a_0+a_1\mathrm{e}^{-\beta\eta}. \tag{14}$$

Substituting the boundary conditions Eq.(6) into Eq.(14), we obtain,

$$a_0=1/\beta,\ a_1=-1/\beta. \tag{15}$$

Substituting Eq.(15) into Eq.(12), we obtain the following values of β for different values of λ:

$$\beta=\sqrt{\frac{1+M}{1-\lambda}},\ \lambda<1. \tag{16}$$

This result is the same as that obtained analytically by Andersson (1992). With f as in Eq.(14), Eq.(11) is now reduced to,

$$(n^2\beta^2-n\sigma a_0\beta+\sigma\alpha)b_n=\sigma\beta(n-2)a_1b_{n-1},\ n\geq2. \tag{17}$$

The problem is now reduced to solving Eqs.(8), (9), (10) and (17).

Theorem 1 Let β satisfy Eq.(16). If $n^2\beta^2-n\sigma a_0\beta+\sigma\alpha\neq0$ for $n\geq1$, where $a_0=1/\beta$, then the boundary value problem Eqs.(4)~(6) has no solution of the form Eq.(7).

Proof From Eq.(10), Eq.(17) and the conditions in the theorem, we have b_n=0, $n\geq1$. This implies that $\theta\equiv0$, which does not satisfy the boundary condition Eq.(6). Therefore a necessary condition for the solution θ to exist is that $n^2\beta^2-n\sigma a_0\beta+\sigma\alpha=0$ for some $n\geq1$.

Lemma 1 Let f as given in Eq.(14) be a solution to Eqs.(4) and (6), where a_0 and a_1 are given in Eq.(15) and β satisfies Eq.(16). The following statements then hold:

(a) If

$$n^2\beta^2-n\sigma a_0\beta+\sigma\alpha\begin{cases}=0,n=1;\\ \neq0,n>1,\end{cases}$$

then the solution θ of the form Eq.(7) to Eq.(5) and Eq.(6) is unique and is given by $\theta(\eta)=\mathrm{e}^{-\beta\eta}$.

(b) If

$$n^2\beta^2-n\sigma a_0\beta+\sigma\alpha\begin{cases}=0,n=1,m;\\ \neq0,n\neq1,m,\end{cases}$$

then the solution for θ exists in the form $\theta(\eta)=b_1\mathrm{e}^{-\beta\eta}+\sum_{n=m}^{\infty}b_n\mathrm{e}^{-n\beta\eta}$ and is not unique.

(c) If

$$n^2\beta^2 - n\sigma a_0\beta + \sigma\alpha \begin{cases} = 0, \ n = m; \\ \neq 0, \ n \neq m, \end{cases}$$

then there exists at least one solution θ of the form $\theta(\eta) = \sum_{n=m}^{\infty} b_n \mathrm{e}^{-n\beta\eta}$.

(d) Otherwise the solution of θ of the form Eq.(7) does not exist.

Proof (a) From Eqs.(10), (17) and the condition of the lemma, we get, $b_1\neq 0$ and $b_n=0$ for $n>1$ which implies that $\theta(\eta)=b_1\mathrm{e}^{-\beta\eta}$. If θ satisfies the boundary condition Eq.(6), then $\theta(0)=b_1=1, \theta(\eta)=\mathrm{e}^{-\beta\eta}$.

(b) From Eqs.(10), (17) and the condition (b) of the lemma, it is concluded that b_1 and b_m are arbitrary and all other coefficients i.e. $b_2, b_3, \ldots, b_{m-1}$ will vanish. Hence,

$$\theta(\eta) = b_1\mathrm{e}^{-\beta\eta} + \sum_{n=m}^{\infty} b_n \mathrm{e}^{-n\beta\eta}, \quad (18)$$

and is not unique. Now to show that the above expression for θ converges and satisfies the boundary condition $\theta(0)=1$ [$\theta(\infty)=0$ is obviously satisfied]. Eq.(17) can be written as

$$b_n = \frac{\sigma\beta(n-2)a_1}{n^2\beta^2 - n\sigma a_0\beta + \sigma\alpha} b_{n-1}, \ n > m. \quad (19)$$

Clearly b_n, $n>m$ depends only on b_m. Here, we have shown that the above expression Eq.(18) for $\theta(\eta)$ is convergent by showing that the set of b_m for which $\sum_{n=m}^{\infty} b_n \mathrm{e}^{-n\beta\eta}$ converges is not empty. It is then sufficient to consider the convergence of $\sum_{n=m}^{\infty} |b_n|$. The expression Eq.(19) can be written as

$$b_n = c_{n-1} b_{n-1},$$

where

$$c_{n-1} = \frac{\sigma\beta(n-2)a_1}{n^2\beta^2 - n\sigma a_0\beta + \sigma\alpha}.$$

Then we have

$$b_n = c_{n-1}c_{n-2}\ldots c_m b_m,$$

$$\limsup_{n\to\infty} |b_n / b_{n-1}| = \limsup_{n\to\infty} |c_{n-1}|$$
$$= \limsup_{n\to\infty} \left| \frac{\sigma\beta(n-2)a_1}{n^2\beta^2 - n\sigma a_0\beta + \sigma\alpha} \right| = 0.$$

This implies that $\sum_{n=m}^{\infty} |b_n| (\Rightarrow \sum_{n=m}^{\infty} b_n)$ and hence $\sum_{n=m}^{\infty} b_n \mathrm{e}^{-n\beta\eta}$ converges for any b_m. Putting $\eta=0$ in the expression Eq.(18) we get

$$\theta(0) = b_1 + \sum_{n=m}^{\infty} b_n .$$

We have already pointed out that b_n, $n>m$ depends only on b_m. Hence $b_1 + \sum_{n=m}^{\infty} b_n$ is a continuous function of b_m with range $(-\infty,\infty)$. Then according to the Intermediate value theorem, there exists at least one b_m such that $b_1 + \sum_{n=m}^{\infty} b_n = 1$.

(c) Follows from (b).

(d) Follows immediately from Eq.(17).

Theorem 2 Let the following conditions be satisfied:

(a) $\lambda<1$;

(b) $\sigma(1-\alpha)(1-\lambda) = 1 + M$.

Then the boundary value problem Eqs.(4)~(6) has exactly a unique pair of solutions of the form

$$f(\eta) = a_0 + a_1\mathrm{e}^{-\beta\eta}, \ \theta(\eta) = \mathrm{e}^{-\beta\eta},$$

where a_0, a_1 and β are given by Eqs.(15) and (16) respectively.

Proof Follows immediately from Lemma 1(a).

Theorem 3 Let $\sigma = \frac{(1+M)(1+m)}{1-\lambda}$ and $\alpha = m/(m+1)$. Eqs.(4)~(6) then have a unique solution f of the form Eq.(14), where a_0, a_1 and β satisfy Eqs.(15) and (16) and infinitely many solutions θ of the form

$$\theta(\eta) = b_1\mathrm{e}^{-\beta\eta} + \sum_{n=m}^{\infty} b_n \mathrm{e}^{-n\beta\eta},$$

Proof Follows from Lemma 1(b).

Theorem 4 Let σ and α satisfy $m^2((1+M)/(1-\lambda))-m\sigma+\sigma\alpha=0$ and $n^2((1+M)/(1-\lambda))-n\sigma+\sigma\alpha\neq 0$ for $n\neq m$. Then Eqs.(4)~(6) has a unique solution f of the form Eq.(14), where a_0, a_1 and β satisfy Eqs.(15) and (16) and at least one solution θ of the form

$$\theta(\eta)=\sum_{n=m}^{\infty} b_n \mathrm{e}^{-n\beta\eta}.$$

Proof Follows immediately from Lemma 1(c).

The main question now is about the convergence of the series f and θ in Eq.(7).

Lemma 2 Let

$$A_1=\sup\left\{\frac{|\sigma\beta|}{\left|\beta^2-\dfrac{\sigma a_0\beta}{n}+\dfrac{\sigma\alpha}{n^2}\right|}, n>2\right\},$$

$$A_2=\sup\left\{\frac{1}{\left|\lambda a_0\beta^2-\dfrac{\beta}{n}+\dfrac{a_0}{n^2}+\dfrac{M}{n^2\beta}\right|}, n\geq 2\right\},$$

$$A=\max\{A_1, A_2\},$$

and let

$$|a_1|=B \text{ and } |b_2|\leq AB^2/2^2.$$

Then

$$|a_n|\leq A^{n-1}B^n/n^2 \text{ for } n\geq 1$$

and

$$|b_n|\leq\frac{A^{n-1}B^n}{n^2} \text{ for } n\geq 2.$$

Proof The proof is straightforward and follows from Eqs.(9) and (11) respectively by mathematical induction.

Theorem 5 Suppose that the conditions in Lemma 2 are satisfied. The solutions f and θ to Eqs.(4) and (5) in full expansion as in Eq.(7) then exist if $AB<1$.

Proof From Lemma 2 we have

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=\lim_{n\to\infty}\frac{|b_{n-1}|}{|b_n|}\leq AB<1.$$

This implies the convergence of $f(\eta)$ and $\theta(\eta)$ in Eq.(7).

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