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E-mail: jzus@zju.edu.cn

On generalized extending modules*

ZENG Qing-yi[1,2]

([1]*Department of Mathematics, Zhejiang University, Hangzhou 310027, China*)

([2]*Department of Mathematics, Shaoguan University, Shaoguan 512005, China*)

E-mail: zqy67@163.com

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Abstract: A module M is called generalized extending if for any submodule N of M, there is a direct summand K of M such that $N\leq K$ and K/N is singular. Any extending module and any singular module are generalized extending. Any homomorphic image of a generalized extending module is generalized extending. Any direct sum of a singular (uniform) module and a semi-simple module is generalized extending. A ring R is a right Co-H-ring if and only if all right R modules are generalized extending modules.

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INTRODUCTION

In recent years theory of extending modules and rings has come to play an important role in the theory of rings and modules. A module M is called extending module (or CS module) if every submodule is essential in a direct summand, or equivalently, every closed submodule is a direct summand. Although this generalization of injectivity is extremely useful, it does not satisfy some important properties. For example, direct sums of extending modules need not be extending; also, the homomorphic image of an extending module need not be extending; yet also, a submodule of an extending module need not be extending. Much work has been done on finding necessary and sufficient conditions to ensure that the extending property is preserved under various extensions.

In this paper, let M be a module, we replace the condition "every submodule is essential in a direct summand" of an extending module by "for every submodule N of M there is a direct summand K of M such that K/N is singular". Thus we generalize both extending modules and singular modules.

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In Section 2, we give the definition of generalized extending modules and show that any direct summand of a generalized extending module and any homomorphic image of a generalized extending module are all generalized extending modules.

In Section 3, we discuss when a direct sum of generalized extending modules is a generalized extending module. We show that a direct sum of a singular module and a semi-simple module is generalized extending.

In Section 4, we characterize co-H-rings by generalized extending modules.

Throughout this paper, unless otherwise stated, all rings are associative rings with identity and all modules are unitary right R-modules.

A submodule N of M is called an essential submodule, denoted by $N\leq_e M$, if for any nonzero submodule L of M, $L\cap N\neq 0$. A closed submodule N of M, denoted by $N\leq_c M$, is a submodule which has no proper essential extension in M. If $L\leq_c N$ and $N\leq_c M$, then $L\leq_c M$ (Goodearl, 1976).

In (Goodearl, 1976) a module M is called singular if $Z(M)=M$, where $Z(M)=\{m\in M|mI=0$ for some essential right ideal I of $R\}$ and called nonsingular if $Z(M)=0$. A ring R is called right nonsingular if R_R is nonsingular. It is well known that if N is essential in

M then M/N is singular. The converse holds if M is nonsingular.

Let M be an R-module, we use $Rad(M)$ to denote the Jacobson radical of M and $r(m)=\{r\in R|mr=0\}$ the right annihilator of $m\in M$. First we collect some well-known facts.

Lemma 1 Let R be a ring, then we have the following:

(1) The class of all singular modules is closed under submodules, factor modules, and direct sums.

(2) Suppose that R is right nonsingular. Then the class of all singular modules is closed under module extension and essential extension.

(3) Let M be a nonsingular module and $N\leq M$. Then M/N is singular if and only if $N\leq_e M$.

(4) If $A\leq B\leq C$, then $A\leq_e C$ if and only if $A\leq_e B$ and $B\leq_e C$.

Proof See Propositions 1.22, 1.23 and 1.21, 1.1 in (Goodearl, 1976).

GENERALIZED EXTENDING MODULES

Now we give the definition of generalized extending module, which generalizes extending modules, as follows:

Definition 1 A module M is called generalized extending if for any submodule N of M, there is a direct summand K of M such that $N\leq K$ and that K/N is singular.

As was mentioned, any extending module is generalized extending and any singular module is generalized extending. But in general, generalized extending module need not be extending.

Example 1 Let M be a (singular) R-module with a unique composition $M\supset U\supset V\supset 0$. From (Dung *et al.,* 1994), $M\oplus U/V$ is not extending but is a generalized extending module.

Proof Let $S=U/V$, S is simple. Let N be any submodule of $M\oplus U/V$, then $N+M=M\oplus[(N+M)\cap S]$. Since S is simple, so $N+M$ is a direct summand of $M\oplus U/V$. Because $(N+M)/N\cong M/(N\cap M)$, $(N+M)/N$ is singular, since M is uniform. Thus, $M\oplus U/V$ is generalized extending.

Also, generalized extending modules need not be singular.

Example 2 Let $\mathbb{Z}$ be the ring of all integers. Then $\mathbb{Z}$ is extending and thus is generalized extending as a right $\mathbb{Z}$-module. But $\mathbb{Z}$ is nonsingular.

The closed submodule of an extending module is a direct summand. But for generalized extending modules, the closed submodule may not be a direct summand. For example, let $M=\mathbb{Z}_p\oplus\mathbb{Z}_{p^3}$ be a $\mathbb{Z}$-module, where p is a prime integer. Obviously, M is generalized extending, but $N=\mathbb{Z}(1+\mathbb{Z}_p;\ p+\mathbb{Z}_{p^3})$ is closed and is not a direct summand.

The following proposition is an easy consequence of Definition 1:

Proposition 1 The following are equivalent for a module M:

(1) M is generalized extending;

(2) For any submodule N of M, M has a decomposition $M=K\oplus K'$ such that $N\leq K$ and that $M/(K'+N)$ is singular;

(3) For any submodule N of M, M/N has a decomposition $M/N=K/N\oplus K'/N$ such that K is a direct summand of M and that M/K' is singular;

(4) For any submodule N of M, there is a direct summand K of M such that for any $x\in K$ there is an essential right ideal I of R such that $xI\subseteq N$.

If M is projective and N is a submodule of M, then M/N is singular if and only if $N\leq_e M$. So, we have:

Proposition 2 Let M be a nonsingular module or projective. Then M is extending if and only if M is generalized extending.

Any submodule of a singular module is singular, while a submodule of an extending module need not be extending. Submodule of a generalized extending module may not be generalized extending.

Example 3 Let $R=\begin{pmatrix}\mathbb{Z} & \mathbb{Z}\\ 0 & \mathbb{Z}\end{pmatrix}$. By (Chatters and Hajarnavis, 1977), R is not right extending, hence it is not a generalized extending module, since R is right nonsingular. But R_R is a submodule of its injective hull S_R, while S_R is an extending module (hence a generalized extending module).

But some special submodules of a generalized extending module may be generalized extending modules.

A submodule N of M is called a fully invariant submodule if for every $f\in S$, we have $f(N)\subseteq N$, where $S=End_R(M)$. If $M=K\oplus L$ and N is a fully invariant submodule of M, we have $N=(N\cap K)\oplus(N\cap L)$ and $M/N\cong K/(N\cap K)\oplus L/(N\cap L)$.

Proposition 3 Let M be a generalized extending module. Then any fully invariant submodule is generalized extending.

Proof Let N be a fully invariant submodule of M and L be a submodule of N. Then, since M is generalized extending, there are direct summands K, K' of M such that $L\leq K$ and K/L is singular. Since N is fully invariant, then $N=(N\cap K)\oplus(N\cap K')$. Obviously, $L\leq N\cap K$ and $(N\cap K)/L\leq K/L$ is singular. So N is generalized extending.

A module M is called distributive if the lattice of submodules of M is distributive.

Corollary 1 Let M be a distributive generalized extending module. Then any submodule is generalized extending.

Proposition 4 Let R be any ring and M an R-module and N a submodule of M.

(1) If M is generalized extending and the intersection of N with any direct summand of M is a direct summand of N, then N is generalized extending;

(2) If N is generalized extending and D a direct summand of M such that $(D+N)/D$ is nonsingular, then $D\cap N$ is a direct summand of N;

(3) If M is generalized extending and $(D+N)/D$ is nonsingular for any direct summand D of M, then N is generalized extending if and only if $D\cap N$ is a direct summand of N for any direct summand D of M.

Proof

(1) It is similar to the proof of Proposition 3.

(2) Let $Y=D\cap N$. Since N is generalized extending, there is a direct summand K of N such that K/Y is singular. If $K\neq Y$, then $D\neq D+K$. For any $d+k\in D+K$ such that $d+k\notin D$, then $k\neq 0$. There is an essential right ideal I of R such that $kI\subseteq Y$. So $(d+k)I\subseteq D$ and $(D+K)/D$ is singular. Since $(D+N)/D$ is nonsingular, we have that $D+K=D$, which is a contradiction. So $K=Y$ is a direct summand of N.

(3) It is a consequence of (1) and (2).

The following proposition shows the equivalent condition of a cyclic submodule of a module to be generalized extending over a right generalized extending ring.

Proposition 5 Let R be a right generalized extending ring and M a cyclic right R-module. Then the following are equivalent:

(1) M is nonsingular;

(2) Every cyclic submodule of M is projective and generalized extending;

(3) Every cyclic submodule of M is projective.

Proof (1)⇒(2) Suppose that M is nonsingular and N a cyclic submodule of M. Then there is a right ideal I of R such that $N\cong R/I$. Since R is generalized extending and N is nonsingular, I is a φ-closed submodule of R_R; hence I is a direct summand of R_R. Thus N is isomorphic to a direct summand of R_R, hence is projective and generalized extending.

(2)⇒(3) It is obvious.

(3)⇒(1) For any $m\in Z(M)$, then mR is projective and is isomorphic to $R/r(m)$, where $r(m)$ is the right annihilator of m. Since R is right generalized extending, then $r(m)$ is a direct summand of R. But $m\in Z(M)$ implies that there is an essential right ideal I of R such that $mI=0$, hence $I\leq r(m)$ and $r(m)\leq_e R$. Thus $r(m)=R$ and $m=0$, hence $Z(M)=0$.

Example 4 (Chatters and Khuri, 1980, Example 2.3)

Let S be the ring of all 3×3 upper triangular matrices over the field of complex numbers and R the subring of S consisting of all elements of S with a real number in the (2; 2)-position. Then R is a right extending ring. Let e be the element of R with 1 in the (3; 3)-position and 0 elsewhere, and set $I=Re$, then I is an ideal of R. But R/I is not a right extending ring.

This example shows that the image of right extending ring need not be right extending ring. But any factor module of singular module is singular and we will show that any image of a generalized extending module is generalized extending.

The direct summand of a extending module is extending (Dung *et al*., 1994). For generalized extending modules, we first show the following proposition and then show that any direct summand of a generalized extending module is generalized extending.

Proposition 6 Let M be a generalized extending module. Then any homomorphic image is generalized extending.

Proof Let $f:M\to N$ be an epimorphism and L a submodule of N. Then there is a submodule H of M such that $L\cong H/Kerf$. Since M is generalized extending, there are direct summands K, K' of M such that $M=K\oplus K'$, $H\leq K$ and that K/H is singular. So $N\cong M/Kerf=(K/Kerf)\oplus(K'+Kerf)/Kerf$ and $L\cong H/Kerf\leq K/Kerf$. Since $(K/Kerf)/(H/Kerf)\cong K/H$ is singular, N is generalized extending.

Corollary 2 (1) Let M be a generalized extending module. Then any direct summand is generalized extending.

(2) Let M be an extending module. Then any nonsingular homomorphic image is extending.

Proof (1) Let N be a direct summand of M and π: $M\rightarrow N$ the canonical projection. Then it follows from Proposition 6 that N is generalized extending.

(2) It is trivial.

Since any module is a homomorphic image of some projective module, we have:

Corollary 3 The following are equivalent:

(1) Every (resp., finitely generated) module is generalized extending;

(2) Every (resp., finitely generated) projective module is generalized extending.

Proposition 7 Let R be a right nonsingular ring and $f{:}M\rightarrow M'$ an epimorphism. Suppose that M' is generalized extending and $Kerf$ is singular injective, then M is generalized extending.

Proof Let N be a submodule of M. First, we assume that $Kerf\subseteq N\leq M$, then $f(N)\leq M'$. Since M' is generalized extending, there is a decomposition of M', $M'=K\oplus H$ such that $K/f(N)$ is singular. So $M=f^{-1}(K)+f^{-1}(H)$. Since $Kerf\leq f^{-1}(H)$ and $Kerf$ is injective, then $f^{-1}(H)=T\oplus Kerf$ for some submodule T of $f^{-1}(H)$. Thus $M=f^{-1}(K)+T$. Since $f^{-1}(K)\cap T\subseteq f^{-1}(K)\cap f^{-1}(H)=Kerf$ and $f^{-1}(K)\cap T\subseteq Kerf\cap T=0$, we have $M=f^{-1}(K)\oplus T$ and $N\leq f^{-1}(K)$.

For any $x\in f^{-1}(K)$, then $f(x)\in K$ and there is an essential right ideal I of R such that $f(x)I\subseteq f(N)$. It is easy to see that $xI\subseteq N$ and that $f^{-1}(K)/N$ is singular.

Now we assume that N does not contain $Kerf$. Set $L=N+Kerf$, then $f(L)=f(N)$. As the case above, there is a decomposition of $M=f^{-1}(K)\oplus T$ such that $f^{-1}(K)/L$ is singular. Since $Kerf$ is singular, we have that $(N+Kerf)/N\cong Kerf/(N\cap Kerf)$ is singular. Since R is right nonsingular, we have that $f^{-1}(K)/N$ is singular.

In either case, M is generalized extending.

Proposition 8 Let R be a right nonsingular ring and M a generalized extending module. Then $M=Z(M)\oplus T$ for some extending submodule T of M and T is $Z(M)$-injective.

Proof If $Z(M)=0$ or $Z(M)=M$, it is trivial.

Suppose that $0<Z(M)<M$. Since M is generalized extending, there are direct summands K, T of M such that $M=K\oplus T$, $Z(M)\leq K$ and that $K/Z(M)$ is singular. So K is singular. Since $Z(M)=Z(K)\oplus Z(T)=K\oplus Z(T)$, so $Z(M)=K$ and T is nonsingular. By Proposition 6, T is extending.

Since for any submodule N of $Z(M)$, $Hom_R(N,T)=0$, so T is $Z(M)$-injective, as required.

Combining with this proposition above, we get the following well-known result about injective modules:

Corollary 4 Let R be a right nonsingular ring and M an injective module. Then $Z(M)$ is injective.

Corollary 5 Let R be a right nonsingular ring and M an indecomposable generalized extending module. Then M is either a singular module or a nonsingular uniform module.

Proposition 9 Let M be a generalized extending module which has maximal submodules. Then for any maximal submodule N of M, either M/N is singular or $M=N\oplus S$ for some simple submodule S of M.

Proof Let N be a maximal submodule of M and suppose that M/N is not singular. Then N is a direct summand of M, i.e., $M=N\oplus S$ for some S of M. Since $S\cong M/N$, so S is simple.

A module M is called local if it has a largest submodule, i.e., a proper submodule which contains all other proper submodules. For a local module M, $Rad(M)$, the Jacobson radical of M is small in M. So we have:

Corollary 6 Let M be a local generalized extending module. Then $M/Rad(M)$ is singular.

Proposition 10 Let R be a right hereditary ring and M an injective module. Then any factor module of M is a direct sum of an injective module and a singular injective module.

Proof Let L be any factor module of M, then there is a submodule N of M such that $L\cong M/N$. Since any injective module is generalized extending, there are direct summands K, K' of M such that $M=K\oplus K'$, $N\leq K$ and that K/N is singular. So $L\cong M/N=K/N\oplus(K'+N)/N$. Since R is hereditary and M is injective, so M/N is injective. Thus K/N is a singular injective module and $(K'+N)/N$ is injective.

DIRECT SUM OF GENERALIZED EXTENDING MODULES

A direct sum of singular modules is also singular. But a direct sum of extending modules may not be extending. Also a direct sum of generalized extending modules need not be generalized extending.

Example 5 (Chatters and Khuri, 1980, Example 2.4)

Let $R=\mathbb{Z}[x]$, where x is an indeterminate and $\mathbb{Z}$ is the ring of integers. The ring R has no proper closed ideals and is extending, hence it is generalized extending. Let $F=R\oplus R$ and set $C=\{(xr,2r)|r\in R\}$. Then C is a closed submodule and not a direct summand of F. Therefore F is not extending. Since $\mathbb{Z}$ is nonsingular, then R is nonsingular and hence F is nonsingular. So F is not generalized extending.

It may be interesting to see when a direct sum of generalized extending modules is generalized extending.

Proposition 11 Let $M=M_1\oplus M_2$ with each M_i, i=1,2 generalized extending. If M is distributive, then M is generalized extending.

Proof Let N be any submodule of M, then $N=(N\cap M_1)\oplus(N\cap M_2)$. Since M_i is generalized extending, there are direct summands H_1, H_1' and H_2, H_2' of M_1, M_2, respectively, such that $M_i=H_i\oplus H_i'$ and $N\cap M_i\leq H_i$ and that $H_i/(N\cap M_i)$ is singular for i=1, 2. Hence $M=(H_1\oplus H_2)\oplus(H_1'\oplus H_2')$ and $N=(N\cap M_1)\oplus(N\cap M_2)\leq(H_1\oplus H_2)$. Since

$$(H_1\oplus H_2)/N=(H_1\oplus H_2)/[(N\cap M_1)\oplus(N\cap M_2)]\\ \cong[H_1/(N\cap M_1)]\oplus[H_2/(N\cap M_2)]$$

is singular, so M is generalized extending.

Corollary 7 Let M be a distributive module and $M=\bigoplus_{i=1}^{n}M_i$. Then M is generalized extending if and only if M_i is generalized extending for all i.

In Example 1, we have shown that $M\oplus U/V$ is generalized extending where M is singular and U/V is simple. In fact, we can generalize this result to the following case:

Theorem 1 Let $M=M_1\oplus M_2$ with M_1 being singular (uniform) and M_2 semi-simple. Then M is generalized extending.

Proof Let N be any submodule of M. Then $N+M_1=M_1\oplus[(N+M_1)\cap M_2]$. Since M_2 is semi-simple, then $(N+M_1)\cap M_2$ is a direct summand of M_2 and therefore $N+M_1$ is a direct summand of M. Note that $(N+M_1)/N\cong M_1/(N\cap M_1)$ is singular, since M_1 is singular (uniform). So M is generalized extending.

Proposition 12 Let $M=M_1\oplus M_2$ with M_1 being generalized extending and M_2 semi-simple. Suppose that for any submodule N of M, $N\cap M_1$ is a direct summand of N. Then M is generalized extending.

Proof Let N be any submodule of M. As in Theorem 1, $N+M_1$ is a direct summand of M. By the hypothesis, $N=(N\cap M_1)\oplus K$ for some submodule K of N. Since M_1 is generalized extending, there is a direct summand T of M_1 such that $T/(N\cap M_1)$ is singular. But $N+M_1=(N\cap M_1)+K+M_1=M_1\oplus K$, so

$$(T\oplus K)/N=(T\oplus K)/[(N\cap M_1)\oplus K]\cong T/(N\cap M_1)\oplus K/K$$

is singular. Since $T\oplus K$ is a direct summand of $N+M_1$ and hence a direct summand of M, M is generalized extending.

Proposition 13 Let $M=M_1\oplus M_2$ with M_1 being generalized extending and M_2 injective. Suppose that for any submodule N of M, $N\cap M_2$ is a direct summand of N, then M is generalized extending.

Proof Let N be any submodule of M, by the hypothesis, there is a submodule N' of N such that $N=(N\cap M_2)\oplus N'$. Note that $N'\cap M_2=0$ and hence $(M_2+N')/N'\cong M_2$ an injective module, so there is a submodule M' of M containing N' such that $M/N'=[(M_2+N')/N']\oplus(M'/N')$. Thus it is easy to see that $M=M_2\oplus M'$ and that $M'\cong M/M_2\cong M_1$. Hence M' is generalized extending.

There are direct summands K, K' of M' such that $M'=K\oplus K'$ and that K/N' is singular. Since $N\cap M_2$ is a submodule of an injective module M_2, so there is a direct summand H of M_2 such that $H=(N\cap M_2)$ is singular. Following from the fact that

$$(H\oplus K)/[(N\cap M_2)\oplus N']\cong[H/(N\cap M_2)]\oplus(K/N')$$

and that $H\oplus K$ is a direct summand of M, M is generalized extending.

Proposition 14 Let $M=M_1\oplus M_2$ such that M_1 is generalized extending and M_2 is an injective module. Then M is generalized extending if and only if for every submodule N of M such that $N\cap M_2\neq 0$, there is a direct summand K of M such that K/N is singular.

Proof Suppose that for every submodule N of M such that $N\cap M_2\neq 0$, there is a direct summand K of M such that K/N is singular. Let N be a submodule of M such that $N\cap M_2=0$. Then, since $(M_2+N)/N\cong M_2$ is an injective module, there is a submodule M' of M containing N such that $M/N=(M'/N)\oplus M_2$. It is easy to see that $M=M'\oplus M_2$. Since $M'\cong M/M_2\cong M_1$ is generalized extending, there is a direct summand K of M', hence of M, such that K/N is singular. So M is generalized extending. The converse is obvious.

CO-H-RINGS

In (Oshiro, 1984), a ring R is called a right co-H-ring if every projective right R-module is extending. It is known that a ring R is a right co-H-ring if and only if R is right $\sum$-extending (i.e., any direct sum of R_R is extending). In this section we characterize co-H-rings by generalized extending modules.

Lemma 2 Let R be a ring. A projective R-module M is generalized extending if and only if every factor module of M is a direct sum of a singular module and a projective module.

Proof Suppose that M is generalized extending. Let M' be any factor module of M, then there is a submodule N of M such that $M/N\cong M'$. Since M is generalized extending, then there are direct summands K, K' of M such that $M=K\oplus K'$ and K/N is singular. Thus $M/N=(K/N)\oplus K'$. As M is projective, K' is projective.

Conversely, let N be any submodule of M, then M/N is a direct sum of a singular module and a projective module. We may assume that $M/N=S/N\oplus T/N$, where S/N is singular and T/N is projective. Then $M=S+T$ and as $M/S\cong T/N$ is projective, S is a direct summand of M. Thus M is generalized extending.

Lemma 3 Let R be any right nonsingular ring. Then the following are equivalent:

(1) All modules are generalized extending;

(2) All projective modules are generalized extending;

(3) All nonsingular modules are extending.

Proof (1)⇔(2) This is Corollary 3.

(1)⇔(3) This is a consequence of Propositions 2, 6, since in a right nonsingular ring R all projective modules are nonsingular.

As an immediate consequence of Lemmas 2, 3 above and of Proposition 2, we have:

Theorem 2 Let R be any ring, then the following are equivalent:

(1) R is a right co-H-ring;

(2) All right R-modules are generalized extending;

(3) All projective right R-modules are extending;

(4) All projective right R-modules are generalized extending;

(5) Every factor module of any projective module is a direct sum of a singular module and a projective module.

Theorem 3 Let R be a right nonsingular ring, then the following are equivalent:

(1) R is a right co-H-ring;

(2) Every nonsingular module is projective;

(3) Every module is generalized extending;

(4) Every nonsingular module is extending;

(5) R is right (and left) Artinian, right (and left) hereditary, right (and left) serial;

(6) $End_R(P)$ is a right nonsingular right extending for each projective module P.

Proof (1)⇒(2) Suppose that R is a right co-H-ring and M a nonsingular module. Then there is a projective module P and an epimorphism f:P→M. Set $K=Kerf$, then K is a closed submodule of P. Since P is extending, then K is a direct summand of P and hence M is isomorphic to a direct summand of P. Thus M is projective.

(2)⇒(4) Let M be a nonsingular module and N a closed submodule of M. Then M/N is nonsingular and, by (2), is projective. So N is a direct summand of M and hence M is extending.

(4)⇒(1) Since a ring R is right nonsingular if and only if all projective modules are nonsingular; by (4), all projective modules are extending and R is a right co-H-ring.

(1)⇔(3) By Theorem 2.

(2)⇔(5)⇔(6) See Theorem 4.2 in (Chatters and Khuri, 1980).

Combining with Theorem 5.23 in (Goodearl, 1976) and Theorem 3 above, we have:

Corollary 8 Let R be any ring, then the following are equivalent:

(1) R is a right nonsingular right co-H-ring;

(2) R is a left nonsingular left co-H-ring;

(3) R is right (and left) hereditary, right (and left) Artinian, and the maximal right and left quotient rings of R coincide.

From (Zeng and Shi, 2006), a module M is called closed weak supplemented if for any closed submodule N of M there is a submodule K of M such that $M=K+N$ and $K\cap N<<M$. A ring R is called a right V-ring if every simple module is injective. Combining with Theorem 3 in (Zeng and Shi, 2006) and Theorem 3 above, we have:

Corollary 9 Let R be a right nonsingular right V-ring, then the following are equivalent:

(1) Every nonsingular module M is closed weak supplemented;

(2) Every nonsingular module M is extending;

(3) Every nonsingular module M is projective;

(4) Every nonsingular module M is a generalized extending module;

(5) R is a right co-H-ring.

Corollary 10 Let R be a ring such that all singular modules are projective, then R is a right co-H-ring if and only if R is semi-simple.

Proof Suppose that R is a right co-H-ring. Let M be any module and N any submodule of M, then by Theorem 2 M is generalized extending, i.e., there is a direct summand K of M such that $N \leq K$ and K/N is singular. By hypothesis K/N is projective, so N is a direct summand of K and hence a direct summand of M. Thus M is semi-simple and R is semi-simple.

The converse is obvious.

A ring R is called a QF-ring if R is left (and right) Artinian and left (and right) self-injective. It is known from Theorem 24.20 in (Faith, 1976) that a ring R is a QF-ring if and only if all projective modules are injective if and only if all injective modules are projective. Obviously every QF-ring R is a left and right co-H-ring. As an immediate consequence of Theorem 3, we have:

Corollary 11 Let R be a right nonsingular ring such that all injective modules are nonsingular. Then R is a right co-H-ring if and only if R is a QF-ring.

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