Journal of Zhejiang University SCIENCE A
ISSN 1673-565X (Print); ISSN 1862-1775 (Online)
www.zju.edu.cn/jzus; www.springerlink.com
E-mail: jzus@zju.edu.cn

On some projectively flat polynomial (*α*,*β*)-metrics

ZHAO Li-li[1,2]

([1]Department of Mathematics, Zhejiang University, Hangzhou 310027, China)

([2]Department of Mathematics, Shanghai Jiao Tong University, Shanghai 200240, China)

E-mail: shally523@163.com

Received Oct. 13, 2006; revision accepted Jan. 16, 2007

Abstract: In this paper, we consider some polynomial (α,β)-metrics, and discuss the sufficient and necessary conditions for a Finsler metric in the form $F=\alpha+a_1\beta+a_2\beta^2/\alpha+a_4\beta^4/\alpha^3$ to be projectively flat, where a_i (i=1,2,4) are constants with $a_1\neq0$, α is a Riemannian metric and β is a 1-form. By analyzing the geodesic coefficients and the divisibility of certain polynomials, we obtain that there are only five projectively flat cases for metrics of this type. This gives a classification for such kind of Finsler metrics.

Key words: Finsler metric, Polynomial, Projectively flat
doi:10.1631/jzus.2007.A0957 **Document code:** A **CLC number:** O186.1

INTRODUCTION

As we know, the flag curvature in Finsler geometry is a natural extension of the sectional curvature in Riemannian geometry. Unlike the sectional curvature, the flag curvature $K=K(P,y)$ of a manifold (M,F) depends on not only the section $P\subset T_xM$ but also a direction y within P. It is one of the fundamental problems to study and characterize Finsler metrics of scalar flag curvature, namely, the flag curvature $K=K(x,y)$ is a scalar function on TM. It is well known that a Riemannian metric is of scalar flag curvature if and only if it has isotropic sectional curvature $K=K(x)$ (=constant in dimension $n>2$ by the Schur Lemma). It is also well known that a Riemannian metric is locally projectively flat if and only if it is of constant sectional curvature (Beltrami theorem). However, this is not the case for the flag curvature of Finsler metrics. There are many Finsler metrics of scalar flag curvature, but not of constant flag curvature. We know that every locally projectively flat Finsler metric has scalar flag curvature (Shen, 2004). Thus locally projectively flat Finsler metrics form a rich class of Finsler metrics of scalar flag curvature. On the other hand, the Hilbert Fourth Problem in the regular case is on the study and characterization of projectively flat Finsler metrics on a convex domain in $\mathbb{R}^n$. Therefore, it is important to study locally projectively flat Finsler metrics.

An important class of Finsler metrics are (α,β)-metrics defined by a Riemannian metric $\alpha=[a_{ij}(x)y^iy^j]^{1/2}$ and a 1-form $\beta=b_i(x)y^i$ in the form $F=\alpha\varphi(s)$, $s=\beta/\alpha$. Where $\varphi=\varphi(s)$ is a positive C^∞ on some open interval $(-b_o,b_o)$ satisfying certain condition. If we take $\varphi=1+s$, we get the so-called Randers metric $F=\alpha+\beta$. If we take $\varphi=1+\varepsilon s+s^2$, we obtain another interesting metric $F=\alpha+\varepsilon\beta+\beta^2/\alpha$. If we take $\varphi=1+\varepsilon s+2s^2-s^4/3$, we obtain $F=\alpha+\varepsilon\beta+2\beta^2/\alpha-\beta^4/(3\alpha^3)$. We know that a Randers metric $F=\alpha+\beta$ is locally projectively flat if and only if α is locally projectively flat and β is closed (Shen, 2003). This is due to (Bácsó and Matsumoto, 1997). The other two metrics are studied in (Shen and Civi Yildirim, 2005; Mo *et al*., 2006; Shen and Zhao, 2006).

In this paper, we shall consider the following class of (α,β)-metrics

$$F=\alpha+a_1\beta+a_2\beta^2/\alpha+a_4\beta^4/\alpha^3,$$

where a_i (i=1,2,4) are constants and $a_1\neq0$. Clearly, this class contains all the above-mentioned metrics. And we prove the following main theorem:

Theorem 1 Let F be a Finsler metric of the form $F=\alpha+a_1\beta+a_2\beta^2/\alpha+a_4\beta^4/\alpha^3$, where α is a Riemannian metric, β is a 1-form, and a_i (i=1,2,4) are constants with $a_1\neq 0$. Then F is projectively flat if and only if one of the following cases occurs:

(1) $a_2=a_4=0$, α is projectively flat and β is a closed 1-form.

(2) $a_2\neq 0$, $a_4=0$ and

(i) $b_{i|j}=\tau_2[(a_2^{-1}+2b^2)a_{ij}-3b_ib_j]$,

(ii) $G_\alpha^i=\eta y^i-\tau_2\alpha^2 b^i$,

where $\tau_2=\tau_2(x)$, $\eta=\eta_i(x)y^i$, b is the norm of β with respect to α.

(3) $a_2=2k$, $a_4=-k^2/3$, where k is a nonzero constant and

(i) $b_{i|j}=\tau_3[(1+4kb^2)a_{ij}-5kb_ib_j]$,

(ii) $G_\alpha^i=\zeta y^i-2k\tau_3\alpha^2 b^i$,

where $\tau_3=\tau_3(x)$, $\zeta=\zeta_i(x)y^i$.

(4) $a_4>0$, α is projectively flat, β is parallel w.r.t. α and the norm $b^2=(\sqrt{a_2^2+12a_4}-a_2)/(6a_4)$.

(5) $a_4<0$, $a_2>0$, $a_2^2+12a_4>0$, α is projectively flat, β is parallel w.r.t. α and $b^2=(\pm\sqrt{a_2^2+12a_4}-a_2)/(6a_4)$.

This main theorem gives a classification for such kind of Finsler metrics.

(α,β)-METRICS

By definition, an (α,β)-metric is expressed in the form $F=\alpha\varphi(s)$, $s=\beta/\alpha$, where $\alpha=(a_{ij}y_iy_j)^{1/2}$ is a Riemannian metric, $\beta=b_i(x)y^i$ a 1-form, and $\varphi=\varphi(s)$ a positive smooth function on an open interval $(-b_o,b_o)$ satisfying

$$\varphi(s)-s\varphi'(s)+(b^2-s^2)\varphi''(s)>0,\ \forall |s|\leq b<b_o.$$

It is known that F is a Finsler metric if and only if $\|\beta\|_\alpha<b_o$ (Hilbert, 1902).

Let $b_{i|j}$ denote covariant derivative of β with respect to α. And define

$$r_{ij}=(b_{i|j}+b_{j|i})/2,\ s_{ij}=(b_{i|j}-b_{j|i})/2,$$
$$s_i=b^js_{ij},\ s_j^i=a^{ik}s_{kj},$$

and

$$r_{00}=r_{ij}y^iy^j,\ s_0=s_iy^i,\ s_0^i=s_j^iy^j,\ s_{i0}=s_{ij}y^j.$$

Clearly, β is closed if and only if $s_{ij}=0$.

The spray coefficients of F and α, denoted by G^i and G_α^i respectively, are given by

$$G^i=\frac{g^{il}}{4}\{[F^2]_{x^ky^l}y^k-[F^2]_{x^l}\},$$
$$G_\alpha^i=\frac{a^{il}}{4}\{[\alpha^2]_{x^ky^l}y^k-[\alpha^2]_{x^l}\},$$

where $(g^{ij}):=(g_{ij})^{-1}$, $g_{ij}:=[F^2]_{y^iy^j}/2$ and $(a^{ij}):=(a_{ij})^{-1}$.

Here and from now on, $[F^2]_{y^i}$, $F_{x^iy^j}$ respectively mean $\partial F^2/\partial y^i$, $\partial^2F/(\partial x^i\partial y^j)$, etc. For the relation between G and G_α, we have the following

Lemma 1 (Shen and Civi Yildirim, 2005) The spray coefficients G^i are related to G_α^i by

$$G^i=G_\alpha^i+\alpha Qs_0^i+J\{-2Q\alpha s_0+r_{00}\}y^i/\alpha$$
$$+H\{-2Q\alpha s_0+r_{00}\}\{b^i-sy^i/\alpha\}, \tag{1}$$

where

$$Q:=\frac{\varphi'}{\varphi-s\varphi'},\quad J:=\frac{\varphi'(\varphi-s\varphi')}{2\varphi[(\varphi-s\varphi')+(b^2-s^2)\varphi'']},$$
$$H:=\frac{\varphi''}{2[(\varphi-s\varphi')+(b^2-s^2)\varphi'']},$$

where $s=\beta/\alpha$ and $b=\|\beta_x\|_\alpha$.

Eq.(1) is given in (Shen, 2004; Chern and Shen, 2005), and a different version can be found in (Kitayama *et al*., 1995; Matsumoto, 1998).

It is well known that a Finsler metric $F=F(x,y)$ on an open subset $U\subset\mathbb{R}^n$ is projectively flat if and only if

$$F_{x^ky^l}y^k-F_{x^l}=0. \tag{2}$$

This is due to (Hamel, 1903). And according to this result, it is obtained that

Lemma 2 (Shen and Civi Yildirim, 2005) An (α,β)-metric $F=\alpha\varphi(s)$, where $s=\beta/\alpha$, is projectively flat on an open subset $U\subset\mathbb{R}^n$ if and only if

$$(a_{ml}\alpha^2-y_my_l)G_\alpha^m+\alpha^3Qs_{l0}$$
$$+H\alpha(-2\alpha Qs_0+r_{00})(b_l\alpha-sy_l)=0. \tag{3}$$

Another important property of G_α is:

Lemma 3 (Yu and You, 2006) If $(a_{ml}\alpha^2-y_m y_l)G_\alpha^m=0$, then α is locally projectively flat.

POLYNOMIAL (α,β)-METRICS

In this section, we consider a special kind of (α,β)-metrics which are called polynomial (α,β)-metrics. A Finsler metric is called a polynomial (α,β)-metric if it has the form

$$F=\alpha\left(1+\sum_{i=1}^{n}a_i s^i\right)=\alpha+\sum_{i=1}^{n}\frac{a_i\beta^i}{\alpha^{i-1}},$$

where n and a_i (i=1,…,n) are constants. If we take n=1 and a_1=1, then the metric is a Randers metric; if we allow n to go to infinity and take all a_i equal to 1, then the resulting metric is the Matsumoto metric. The case $n=a_1=2$ and $a_2=1$ was studied by Shen and Civi Yildirim (2005); Senarath and Thornley (2004) studied the case $n=2$, $a_1=0$ and $a_2=1$; Shen and Zhao (2006) studied such metrics with $a_2=2k$, $a_3=0$ and $a_4=-k^2/3$.

Next, we will classify the projectively flat polynomial (α,β)-metrics of form

$$F=\alpha(1+a_1 s+a_2 s^2+a_4 s^4), \tag{4}$$

where a_1, a_2 and a_4 are constants with $a_1\neq 0$. And according to the following discussion, Theorem 1 will be proved.

Let $b_o=b_o(a_1,a_2,a_4)>0$ be the largest number such that $\forall |s|\leq b<b_o$,

$$\begin{cases}1+a_1 s+a_2 s^2+a_4 s^4>0,\\ (1+2b^2a_2)+(12b^2a_4-3a_2)s^2-15a_4s^4>0,\end{cases} \tag{5}$$

then F is a Finsler metric if and only if $b=\|\beta_x\|_\alpha<b_o$ for any point x of the manifold. From now on, we always assume that a_1, a_2 and a_4 satisfy Eq.(5).

In order to see under what condition F of Eq.(4) is projectively flat, we should apply Lemma 2. First we compute the coefficients

$$Q=\frac{a_1+2a_2s+4a_4s^3}{1-a_2s^2-3a_4s^4},$$

$$J=\frac{1}{2}\frac{a_1+2a_2s+4a_4s^3}{1+a_1s+a_2s^2+a_4s^4}\cdot\frac{1-a_2s^2-3a_4s^4}{1-3a_2s^2-15a_4s^4+2b^2a_2+12b^2a_4s^2},$$

$$H=\frac{a_2+6a_4s^2}{1-3a_2s^2-15a_4s^4+2b^2a_2+12b^2a_4s^2}.$$

Then Eq.(3) is reduced to the following one

$$\begin{aligned}&(a_{ml}\alpha^2-y_my_l)G_\alpha^m+\frac{(a_1+2a_2s+4a_4s^3)\alpha^3s_{l0}}{1-a_2s^2-3a_4s^4}\\&+\frac{a_2+6a_4s^2}{1-3a_2s^2-15a_4s^4+2b^2a_2+12b^2a_4s^2}\cdot\\&\left\{-2\alpha\frac{a_1+2a_2s+4a_4s^3}{1-a_2s^2-3a_4s^4}s_0+r_{00}\right\}(b_l\alpha^2-\beta y_l)=0.\end{aligned} \tag{6}$$

First, we rewrite Eq.(6) as a polynomial in y^i and α, which is linear in α. This gives

$$\begin{aligned}0=&(\alpha^4-3a_2\beta^2\alpha^2-15a_4\beta^4+2b^2a_2\alpha^4+12b^2a_4\beta^2\alpha^2)\cdot\\&(\alpha^4-a_2\beta^2\alpha^2-3a_4\beta^4)(a_{ml}\alpha^2-y_my_l)G_\alpha^m\\&+(a_1\alpha^3+2a_2\beta\alpha^2+4a_4\beta^3)\alpha^4s_{l0}\cdot\\&(\alpha^4-3a_2\beta^2\alpha^2-15a_4\beta^4+2b^2a_2\alpha^4+12b^2a_4\beta^2\alpha^2)\\&-2\alpha^4(a_1\alpha^3+2a_2\beta\alpha^2+4a_4\beta^3)s_0(a_2\alpha^2+6a_4\beta^2)(b_l\alpha^2-\beta y_l)\\&+(a_2\alpha^2+6a_4\beta^2)(\alpha^4-a_2\beta^2\alpha^2-3a_4\beta^4)r_{00}\alpha^2(b_l\alpha^2-\beta y_l).\end{aligned} \tag{7}$$

The coefficients of α must be zero (note: α^{even} is a polynomial in y^i). We obtain

$$\begin{aligned}&s_{l0}a_1\alpha^7(\alpha^4-3a_2\beta^2\alpha^2-15a_4\beta^4+2b^2a_2\alpha^4+12b^2a_4\beta^2\alpha^2)\\&=2a_1\alpha^7s_0(a_2\alpha^2+6a_4\beta^2)(b_l\alpha^2-\beta y_l).\end{aligned}$$

Since $a_1\neq 0$, we have

$$\begin{aligned}&s_{l0}(\alpha^4-3a_2\beta^2\alpha^2-15a_4\beta^4+2b^2a_2\alpha^4+12b^2a_4\beta^2\alpha^2)\\&=2s_0(a_2\alpha^2+6a_4\beta^2)(b_l\alpha^2-\beta y_l).\end{aligned} \tag{8}$$

Contracting Eq.(8) with b^l yields

$$s_0(\alpha^4-a_2\beta^2\alpha^2-3a_4\beta^4)=0. \tag{9}$$

For $\alpha^4-a_2\beta^2\alpha^2-3a_4\beta^4\neq 0$, we have $s_0=0$. Then it follows from Eqs.(5) and (8) that

$$s_{l0}=0. \tag{10}$$

Thus β is closed.

Now $s_{ij}=0$, then Eq.(7) can be reduced to the following

$$(\alpha^4-3a_2\beta^2\alpha^2-15a_4\beta^4+2b^2a_2\alpha^4+12b^2a_4\beta^2\alpha^2)\cdot(a_{ml}\alpha^2-y_my_l)G_\alpha^m+(a_2\alpha^2+6a_4\beta^2)r_{00}\alpha^2(b_l\alpha^2-\beta y_l)=0. \quad (11)$$

Then, contracting Eq.(11) with b^l, we get

$$(\alpha^4-3a_2\beta^2\alpha^2-15a_4\beta^4+2b^2a_2\alpha^4+12b^2a_4\beta^2\alpha^2)\cdot(b_m\alpha^2-y_m\beta)G_\alpha^m+(a_2\alpha^2+6a_4\beta^2)r_{00}\alpha^2(b^2\alpha^2-\beta^2)=0. \quad (12)$$

We now distinguish several cases.

Case 1 $a_2=a_4=0$

Then, the metric is a Randers metric. It is well known that a Randers metric $F=\alpha+\beta$ is projectively flat if and only if α is projectively flat and β is a closed 1-form, which is case (1) of Theorem 1.

Case 2 $a_2=0$ and $a_4\neq 0$

Then, Eq.(12) is reduced to the following

$$(\alpha^4-15a_4\beta^4+12b^2a_4\beta^2\alpha^2)(b_m\alpha^2-\beta y_m)G_\alpha^m=-6a_4r_{00}\beta^2\alpha^2(b^2\alpha^2-\beta^2). \quad (13)$$

First consider this equation on the open subset where $b\neq 0$. Since $\alpha^4-15a_4\beta^4+12b^2a_4\alpha^2\beta^2$ is not divisible by α^2 or β^2, it must be divisible by $b^2\alpha^2-\beta^2$. Thus, there are two functions c_1 and c_2 such that

$$\alpha^4-15a_4\beta^4+12b^2a_4\alpha^2\beta^2=(b^2\alpha^2-\beta^2)(c_1\alpha^2+c_2\beta^2). \quad (14)$$

We can then get the equation system

$$c_1b^2=1;\ b^2c_2-c_1=12b^2a_4;\ c_2=15a_4.$$

Hence we rewrite this system as

$$c_1=b^{-2},\ c_2=5b^{-4},\ a_4=1/(3b^4).$$

Now, we can see a_4 is a positive number, and c_1, c_2, b are actually constant. Typically, we write $a_4>0$, and $b=(3a_4)^{-1/4}$. By continuity, b is identically zero or should never vanish. The first case is Riemannian which can be covered by Case 1, hence we just consider the second case $b=(3a_4)^{-1/4}$.

Thus, Eq.(13) can be reduced to

$$(\alpha^2/b^2+5\beta^2/b^4)(b_m\alpha^2-\beta y_m)G_\alpha^m=-2\beta^2r_{00}\alpha^2/b^4. \quad (15)$$

Since $\alpha^2/b^2+5\beta^2/b^4$ is not divisible by $2\beta^2/b^4$ and α^2, it must be divisible by r_{00}. Therefore, there must be a scalar function $\tau_1=\tau_1(x)$ such that

$$r_{00}=\tau_1(b^2\alpha^2+5\beta^2). \quad (16)$$

Notice that β is a closed 1-form, we have

$$b_{i|j}=r_{ij}=\tau_1(b^2a_{ij}+5b_ib_j). \quad (17)$$

Contracting with b^i, since b is a constant, we obtain $0=b^ib_{i|j}=6\tau_1b_jb^2$. Because $b>0$, we get $\tau_1=0$ and $r_{00}=0$. From Eq.(17), we have $b_{i|j}=0$, i.e., β is parallel with respect to α. Thus, Eq.(11) is reduced to

$$(b^2\alpha^2-\beta^2)(\alpha^2/b^2+5\beta^2/b^4)(a_{ml}\alpha^2-y_my_l)G_\alpha^m=0. \quad (18)$$

Since $(b^2\alpha^2-\beta^2)(\alpha^2/b^2+5\beta^2/b^4)\neq 0$, we must have $(a_{ml}\alpha^2-y_my_l)G_\alpha^m=0$. From Lemma 3, we have $G_\alpha^i=0$, i.e., α is projectively flat.

Finally, in this case, we have the following result: The Finsler metric $F=\alpha+a_1\beta+a_4\beta^4/\alpha^3$ (a_1, $a_4\neq 0$) is projectively flat if and only if α is projectively flat, β is parallel with respect to α and $a_4=1/(3b^4)$.

This case is contained in case (4) of Theorem 1.

Case 3 $a_2\neq 0$ and $a_4=0$

In this case, the metric is $F=\alpha(1+a_1s+a_2s^2)$, which was studied by Shen and Civi Yildirim (2005). We see that $F=\alpha(1+a_1s+a_2s^2)$ is projectively flat if and only if

(1) $G_\alpha^i=\eta y^i-\tau_2\alpha^2b^i$,

(2) $b_{i|j}=\tau_2[(a_2^{-1}+2b^2)a_{ij}-3b_ib_j]$,

where $\tau_2=\tau_2(x)$, $\eta=\eta_i(x)y^i$ and a_2 is a nonzero constant.

In this case,

$$G^i=(\eta+\tau_2\alpha\chi_3)y^i,\ \chi_3:=\frac{(1-a_2s^2)(a_1+2a_2s)}{2a_2(1+a_1s+a_2s^2)}-a_2s.$$

Hence case (2) in Theorem 1 is true.

Case 4 $a_2\neq 0$ and $a_4\neq 0$

In this case, recall Eq.(12),

$$\alpha^4-3a_2\beta^2\alpha^2-15a_4\beta^4+2b^2a_2\alpha^4+12b^2a_4\beta^2\alpha^2 \quad (19)$$

is divisible by $(a_2\alpha^2+6a_4\beta^2)$ or $(b^2\alpha^2-\beta^2)$ at each point. Again, we consider them when $b\neq 0$. We now have to distinguish two further cases.

Case (i) Expression (19) is divisible by $(a_2\alpha^2+6a_4\beta^2)$ at some point.

Then, there are two numbers c_3 and c_4 such that

$$\alpha^4-3a_2\beta^2\alpha^2-15a_4\beta^4+2b^2a_2\alpha^4+12b^2a_4\beta^2\alpha^2 =(a_2\alpha^2+6a_4\beta^2)(c_3\alpha^2+c_4\beta^2). \quad (20)$$

We can get the equation system

$$\begin{cases} c_3a_2 = 1+2b^2a_2, \\ 6c_3a_4+c_4a_2 = 12b^2a_4-3a_2, \\ 6c_4a_4 = -15a_4. \end{cases}$$

Solving this system, since $a_2\neq 0$ and $a_4\neq 0$, we get

$$c_3 = 2b^2+a_2^{-1},\ c_4=-5/2,\ a_2^2=-12a_4.$$

Without loss of generality, we take $a_2=2k$ and $a_4=-k^2/3$, then the metric can be expressed as

$$F=\alpha+a_1\beta+2k\beta^2/\alpha-k^2\beta^4/(3\alpha^3),$$

where k is a nonzero real number. Fortunately, this case has been studied by Shen and Zhao (2006), and the result is as follows:

Proposition 1 (Shen and Zhao, 2006) The Finsler metric $F=\alpha+a_1\beta+2k\beta^2/\alpha-k^2\beta^4/(3\alpha^3)$ $(k\neq 0)$ is projectively flat if and only if

(1) $G_\alpha^i = \zeta y^i - 2k\tau_3\alpha^2 b^i$,

(2) $b_{i|j}=\tau_3[(1+4kb^2)a_i-5kb_ib_j]$,

where $\tau_3=\tau_3(x)$, $\zeta=\zeta_i(x)y^i$.

Now, this proposition ends the discussion of this case which is just case (3) of Theorem 1.

Case (ii) Expression (19) is divisible by $(b^2\alpha^2-\beta^2)$ at each point.

Similarly, we have two functions c_5 and c_6 such that

$$\alpha^4-3a_2\beta^2\alpha^2-15a_4\beta^4+2b^2a_2\alpha^4+12b^2a_4\beta^2\alpha^2 =(b^2\alpha^2-\beta^2)(c_5\alpha^2+c_6\beta^2). \quad (21)$$

We get the following system

$$\begin{cases} c_5b^2 = 1+2b^2a_2, \\ b^2c_6-c_5 = 12b^2a_4-3a_2, \\ c_6 = 15a_4. \end{cases}$$

Rewriting the system, we have

$$\begin{cases} c_5 = (1+2b^2a_2)/b^2, \\ c_6 = 5(1-b^2a_2)/b^4, \\ a_4 = (1-b^2a_2)/(3b^4). \end{cases}$$

This implies c_5, c_6 and b are all constant. Again, we know b is identically zero or never vanishes as in Case 2. Moreover, the third equation means $3a_4b^4+a_2b^2-1=0$ and the only possible solutions are

(1) $a_4>0$, and $b=\left[\left(\sqrt{a_2^2+12a_4}-a_2\right)\Big/(6a_4)\right]^{1/2}$,

(2) $a_4<0$, $a_2>0$, $a_2^2+12a_4\geq 0$ and

$$b=\left[\left(\pm\sqrt{a_2^2+12a_4}-a_2\right)\Big/(6a_4)\right]^{1/2}.$$

Since $a_2^2+12a_4=0$ is just case (1), we can drop it.

Now, substituting Eq.(21) and c_5, c_6 and a_4 into Eq.(12), one can get

$$\{b^2(1+2b^2a_2)\alpha^2+5(1-b^2a_2)\beta^2\}(b_m\alpha^2-\beta y_m)G_\alpha^m = -\{a_2b^4\alpha^2+2(1-b^2a_2)\beta^2\}r_{00}\alpha^2. \quad (22)$$

If $b^2(1+2b^2a_2)\alpha^2+5(1-b^2a_2)\beta^2$ is divisible by $a_2b^4\alpha^2+2(1-b^2a_2)\beta^2$, then

$$\frac{b^2(1+2b^2a_2)}{a_2b^4}=\frac{5(1-b^2a_2)}{2(1-b^2a_2)}.$$

Solving this equation, we have $a_2=2/b^2$. In this case, $a_4=-1/(3b^4)$, so that

$$F=\alpha[1+a_1s+2s^2/b^2-s^4/(3b^4)].$$

From Lemma 4.1 in (Shen and Zhao, 2006), we know that it is not a Finsler metric which leads to a contradiction. Thus, $a_2\neq 2/b^2$ and $b^2(1+2b^2a_2)\alpha^2+5(1-b^2a_2)\beta^2$ is not divisible by $a_2b^4\alpha^2+2(1-b^2a_2)\beta^2$.

Because $b^2(1+2b^2a_2)\alpha^2+5(1-b^2a_2)\beta^2$ is not divisible by α^2, $(b_m\alpha^2-\beta y_m)G_\alpha^m$ must be divisible by $\alpha^2[a_2b^4\alpha^2+2(1-b^2a_2)\beta^2]$. Therefore, there is a scalar function $\tau_4=\tau_4(x)$ such that

$$r_{00}=\tau_4\{b^2(1+2b^2a_2)\alpha^2+5(1-b^2a_2)\beta^2\}. \quad (23)$$

Note that β is closed, so

$$b_{i|j}=r_{ij}=\tau_4\{b^2(1+2b^2a_2)a_{ij}+5(1-b^2a_2)b_ib_j\}. \quad (24)$$

Contracting with b^i, since b is a constant, we have

$$0=b^ib_{i|j}=(6-3b^2a_2)\tau_4b^2b_j.$$

Since $6-3b^2a_2\neq0$, we have $\tau_4=0$. From Eq.(24), we have $b_{i|j}=0$, i.e., β is parallel with respect to α. Thus Eq.(11) is reduced to

$$\begin{aligned}&\{b^2(1+2b^2a_2)\alpha^2+5(1-b^2a_2)\beta^2\}\cdot\\&(b^2\alpha^2-\beta^2)(a_{ml}\alpha^2-y_my_l)G_\alpha^m=0.\end{aligned} \quad (25)$$

Since $\{b^2(1+2b^2a_2)\alpha^2+5(1-b^2a_2)\beta^2\}(b^2\alpha^2-\beta^2)\neq0$, we have $(a_{ml}\alpha^2-y_my_l)G_\alpha^m=0$. From Lemma 3, we get $G_\alpha^i=0$, i.e., α is projectively flat.

In this case, we have the following conclusion: The metric $F=\alpha+a_1\beta+a_2\beta^2/\alpha+a_4\beta^4/\alpha^3$ is projectively flat if and only if α is projectively flat, β is parallel with respect to α, and b is a positive constant satisfying $3a_4b^4+a_2b^2-1=0$, where a_2 and a_4 satisfy $a_4>0$, $a_2\neq0$ or $a_4<0$, $a_2>0$ and $a_2^2+12a_4>0$.

Then combining with Case 2, cases (4) and (5) of Theorem 1 have been proved.

ACKNOWLEDGEMENTS

The author is grateful to Professor Yibing Shen and Professor Zhongmin Shen for providing helpful comments which led to a significant improvement of this article.

References

Bácsó, S, Matsumoto, M., 1997. On Finsler spaces of Douglas type—a generalization of the notion of Berwald space. *Publ. Math. Debrecen*, **325**:385-406.

Chern, S.S., Shen, Z., 2005. Riemann-Finsler Geometry. World Scientific, Singapore, p.60-73.

Hamel, G., 1903. Über die geometrieen in denen die Geraden die Kürzesten sind. *Math. Ann.*, **57**:231-264. [doi:10.1007/BF01444348]

Hilbert, D., 2001. Mathematical Problems. *Bull. Amer. Math. Soc.*, **37**:407-436. Reprinted from *Bull. Amer. Math. Soc.*, **8**(July 1902):437-479.

Kitayama, M., Azuma, M., Matsumoto, M., 1995. On Finsler spaces with (α,β)-metric, regularity, geodesics and main scalar. *J. Hokkaido Univ. Education*, **46**(1):1-10.

Matsumoto, M., 1998. Finsler spaces with (α,β)-metric of Douglas type. *Tensor, N.S.*, **60**:123-134.

Mo, X., Shen, Z., Yang, C., 2006. Some constructions of projectively flat Finsler metrics. *Sci. in China (Ser. A)*, **49**(5):703-714. [doi:10.1007/s11425-006-0703-7]

Senarath, P., Thornley, G.M., 2004. Locally projectively flat Finsler spaces with (α,β)-metrics. Manuscript.

Shen, Z., 2003. Projectively flat Randers metrics of constant flag curvature. *Math. Ann.*, **325**:19-30. [doi:10.1007/s00208-002-0361-1]

Shen, Z., 2004. Landsberg Curvature, S-curvature and Riemann Curvature, in a Sampler of Riemann-Finsler Geometry. MSRI Series, Vol. 50, Cambridge University Press.

Shen, Z., Civi Yildirim, G., 2005. On a class of projectively flat metrics of constant flag curvature. *Can. J. Math.*, in press.

Shen, Y.B., Zhao, L.L., 2006. Some projectively flat (α,β)-metrics. *Sci. in China (Ser. A)*, **49**(6):838-851. [doi:10.1007/s11425-006-0838-6]

Yu, Y.Y., You, Y., 2006. Projectively flat exponential Finsler metrics. *J. Zhejiang Univ. Sci. A*, **7**(6):1068-1076. [doi:10.1631/jzus.2006.A1068]