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Projectively flat Asanov Finsler metric*

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Abstract: In this work, we study the Asanov Finsler metric $F=\alpha(\beta^2/\alpha^2+g\beta/\alpha+1)^{1/2}\exp\{(G/2)\arctan[\beta/(h\alpha)+G/2]\}$, where $\alpha=(a_{ij}y^iy^j)^{1/2}$ is a Riemannian metric and $\beta=b_iy^i$ is a 1-form, $g\in(-2,2)$, $h=(1-g^2/4)^{1/2}$, $G=g/h$. We give the necessary and sufficient condition for Asanov metric to be locally projectively flat, i.e., α is projectively flat and β is parallel with respect to α. Moreover, we proved that the Douglas tensor of Asanov Finsler metric vanishes if and only if β is parallel with respect to α.

Key words: Exponential Finsler metric, Projectively flat, (α,β)-metrics, Douglas tensor
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INTRODUCTION

It is the Hilbert's Fourth Problem to characterize the (not-necessarily-reversible) distance functions on an open subset in $\mathbb{R}^n$ such that straight lines are the shortest paths (Hilbert, 1902). Distance functions induced by Finsler metrics are regarded as smooth ones. Thus the Hilbert's Fourth Problem in the smooth case is to characterize Finsler metrics on an open subset in $\mathbb{R}^n$ whose geodesics are straight lines. Finsler metrics on an open domain in $\mathbb{R}^n$ with this property are said to be projectively flat. It is important and interesting to study some special projectively flat Finsler metrics such as Randers metric (Shen, 2003), exponential Finsler metric (Yu and You, 2006), etc. Hamel (1903) first found a simple system of partial differential equations (PDEs) to characterize projectively flat Finsler metrics on an open subset in $\mathbb{R}^n$. That is, a Finsler metric $F=F(x,y)$ on an open subset in $\mathbb{R}^n$ is projectively flat if and only if it satisfies the following PDEs:

$$F_{x^ky^l}y^k=F_{x^l}. \tag{1}$$

Shen and Civi Yildirim (2005) studied the locally projectively flat metrics in the form $F=(\alpha+\beta)^2/\alpha$. Senarath and Thornley (2004) gave an equation in local coordinates that characterizes projectively flat Finsler metrics in the form $F=(\alpha^2+\beta^2)/\alpha$. Yu and You (2006) considered the special exponential Finsler, which is in the form $F=\alpha\exp(\beta/\alpha)+\varepsilon\beta$. Shen (2003) observed that a Randers metric $F=\alpha+\beta$ is locally projectively flat if and only if α is locally projectively flat and β is closed. This fact is a direct consequence of Bácsó-Matsumoto's theorem (Bácsó and Matsumoto, 1997) on Doulas metrics. Recently Shen (2006) found a necessary and sufficient condition for the metric to be locally projectively flat in the dimension $n\geq3$. These are some special forms of (α,β)-metric (see Section 2).

Asanov (2005) introduced a very special Finsler type metric in the Minkowski space. This metric is analogous to Randers metric. With this metric, Asanov got a series of interesting results. We find this metric is just a special (α,β)-metric, which has the form

$$F=\alpha(\beta^2/\alpha^2+g\beta/\alpha+1)^{1/2}\exp\{(G/2)\arctan[\beta/(h\alpha)+G/2]\}, \tag{2}$$

where $g\in(-2,2)$, $h=(1-g^2/4)^{1/2}$, $G=g/h$. We call this metric Asanov Finsler metric.

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In this paper, we shall first prove the following

Theorem 1 Let Expression (2) be a Finsler metric on a manifold M. F is locally projectively flat if and only if the following conditions hold:

(1) β is parallel with respect to α;

(2) α is locally projectively flat, i.e., α is of constant curvature.

A theorem due to Douglas states that a Finsler metric F is projectively flat if and only if two special curvature tensors are zero. The first is the Douglas tensor. The second is the projective Weyl tensor for $n\geq3$, and the Berwald-Weyl tensor for $n=2$. It is known that the projective Weyl tensor vanishes if and only if the flag curvature of F has no dependence on the transverse edges (but is possibly dependent on the position x and the flagpole y). If the Douglas tensor of F vanishes, we call F a Douglas metric.

Bácsó and Matsumoto (1997) proved that a Randers metric $F=\alpha+\beta$ is a Douglas metric if and only if β is a closed 1-form. Matsumoto (1998) obtained that for $n=\dim M\geq3$, $F=(\alpha^2+\beta^2)/\alpha$ is a Douglas metric if and only if

$$b_{i|j}=\tau[(1+2b^2)a_{ij}-3b_ib_j],$$

where $\tau=\tau(x)$ is a scalar function.

In this paper, we shall also prove the following

Theorem 2 Let Expression (2) be a Finsler metric on a manifold M. Then the Douglas tensor of F vanishes if and only if β is parallel with respect to α.

(α,β)-METRICS

Finsler metrics under our consideration are special (α,β)-metric, it is expressed in the following form

$$F=\alpha\phi(s),\ s=\beta/\alpha, \tag{3}$$

where $\alpha=(\alpha_{ij}y^iy^j)^{1/2}$ is a Riemannian metric and $\beta=b_iy^j$ is a 1-form. $\phi=\phi(s)$ is a C^∞ positive function on an open interval $(-b_o,b_o)$ satisfying

$$\phi=\phi(s)-s\phi'(s)+(b^2-s^2)\phi''(s)>0,\ \ |s|\leq b<b_o, \tag{4}$$

It is known that F is a Finsler metric if and only if $\|\beta_x\|_\alpha<b_o$ for any $x\in M$ (Chern and Shen, 2005). Let G^i and $G_\alpha{}^i$ denote the spray coefficients of F and α, respectively, given by

$$G^i=g^{il}\{[F^2]_{x^ky^l}y^k-[F^2]_{x^l}\}/4,$$
$$G_\alpha{}^i=a^{il}\{[\alpha^2]_{x^ky^l}y^k-[\alpha^2]_{x^l}\}/4,$$

where $(g_{ij}):=[F^2]_{y^iy^j}/2$ and $(a^{ij}):=(a_{ij})^{-1}$.

Lemma 1 (Chern and Shen, 2005) The geodesic coefficients G^i are related to $G_\alpha{}^i$ by

$$\begin{aligned}G^i=&G_\alpha{}^i+\alpha Qs_0{}^i+J\{-2Q\alpha s_0+r_{00}\}y^i/\alpha\\&+H\{-2Q\alpha s_0+r_{00}\}\{b^i-sy^i/\alpha\},\end{aligned} \tag{5}$$

where

$$Q:=\phi'/(\phi-s\phi'),$$
$$J:=\frac{\phi'(\phi-s\phi')}{2\phi[(\phi-s\phi')+(b^2-s^2)\phi'']},$$
$$H:=\frac{\phi''}{2\phi[(\phi-s\phi')+(b^2-s^2)\phi'']},$$

where $s=\beta/\alpha$, and $b:=\|\beta_x\|_\alpha$, $s_{ij}=(b_{i|j}-b_{j|i})/2$, $s_{l0}=s_{li}y^i$, $s_0=s_{l0}b^l$, $r_{ij}=(b_{i|j}+b_{j|i})/2$ and $r_{00}=r_{ij}y^iy^j$.

Eq.(5) is given in (Chern and Shen, 2005); a different version of Eq.(5) is given in (Matsumoto, 1998).

Lemma 2 (Shen and Civi Yildirim, 2005) An (α,β)-metric $F=\phi(s)$, where $s=\beta/\alpha$ is projectively flat on an open subset $U\subset\mathbb{R}^n$ if and only if

$$\begin{aligned}&(a_{ml}\alpha^2-y_my_l)G_\alpha{}^m+\alpha^3Qs_{l0}\\&+H\alpha\{-2Q\alpha s_0+r_{00}\}\{b_l\alpha-sy_l\}=0,\end{aligned} \tag{6}$$

where $y_m=a_{mi}y^i$.

ASANOV FINSLER METRIC

In this section, we consider the Asanov Finsler metric as given in Expression (2), where $\alpha=(\alpha_{ij}y^iy^j)^{1/2}$ is a Riemannian metric and $\beta=b_iy^j$ is a 1-form on M, $g\in(-2,2)$, $h=(1-g^2/4)^{1/2}$, $G=g/h$. When $g=0$, F is a Riemannian metric.

Let $b_o>0$ be the largest number such that

$$\phi(s)-s\phi'(s)+(b^2-s^2)\phi''(s)>0,\ \ |s|\leq b<b_o,$$

i.e.,

$$(1+gs+b^2)/(s^2+gs+1)^{3/2}>0.$$

Lemma 3 Expression (2) is a Finsler metric.
Proof If Expression (2) is a Finsler metric, then

$$(1+gs+b^2)/(s^2+gs+1)^{3/2}>0.$$

Let $s=b$, and note $-2<g<2$, then $\forall b<b_0$, $1+gb+b^2>0$, thus

$$(1+gb+b^2)/(b^2+gb+1)^{3/2}>0.$$

So Expression (2) is a Finsler metric.

By Lemma 1, we have

$$\begin{cases} Q:=(\beta+g\alpha)/\alpha, \\ H:=\alpha/\{2[g\beta+(1+b^2)\alpha]\}, \\ J:=(\beta+g\alpha)/[g\beta+(1+b^2)\alpha]. \end{cases} \quad (7)$$

Eq.(6) is reduced to the fowlling equation:

$$(a_{ml}\alpha^2-y_m y_l)G_\alpha{}^m+\alpha^2(\beta+g\alpha)s_{l0}+\alpha^2(b_l-\beta y_l)\cdot [-2(\beta+g\alpha)s_0+r_{00}]/2[g\beta+(1+b^2)\alpha^2]=0. \quad (8)$$

Lemma 4 (Yu and You 2006) If $(a_{ml}\alpha^2-y_m y_l)G_\alpha{}^m=0$, then α is locally projectively flat.

By Eq.(8), we can prove Theorem 1.

Proof (of Theorem 1) When g=0, F is a Riemannian metric. Theorem 1 holds.

So in the following discussion we assume $g\neq0$.

If F is projectively flat. We rewrite Eq.(8) as polynomial in y^i and α, which is linear in α. This gives

$$[g\beta+(1+b^2)\alpha](a_{ml}\alpha^2-y_m y_l)G_\alpha{}^m+\alpha^2[g\beta+(1+b^2)\alpha]\cdot(\beta+g\alpha)s_{l0}+[-2\alpha(\beta+g\alpha)s_0+r_{00}](b_l\alpha^2-\beta y_l)/2=0. \quad (9)$$

Contracting Eq.(9) with b^l yields

$$[g\beta+(1+b^2)\alpha](b_m\alpha^2-\beta y_m)G_\alpha{}^m+\alpha[g\alpha^4+(1+g^2)\cdot\alpha^3\beta+2g\alpha^2\beta^2+\alpha\beta^3]s_0+(b^2\alpha^2-\beta^2)\alpha r_{00}/2=0. \quad (10)$$

Because α^{even} is a polynomial in y^i, then the coefficients of α must be zero. We obtain

$$(b_m\alpha^2-\beta y_m)G_\alpha{}^m+(2\beta^2+\alpha^2)\alpha^2=0, \quad (11)$$

$$(1+b^2)(b_m\alpha^2-\beta y_m)G_\alpha{}^m+[(1+g^2)\alpha^2+\beta^2]\beta s_0 +(b^2\alpha^2+\beta^2)r_{00}/2=0. \quad (12)$$

Note that the polynomial β is not divisible by α^2 and $2\beta^2+\alpha^2$. Thus $(b_m\alpha^2-y_m\beta)G_\alpha{}^m$ is divisible by $\alpha^2(2\beta^2+\alpha^2)$. Therefore, there is a scalar function $\tau=\tau(x)$ such that

$$(b_m\alpha^2-\beta y_m)\beta G_\alpha{}^m=-\tau(x)(2\beta^2+\alpha^2)\alpha^2 s_0, \quad (13)$$

$$s_0=\tau(x)\beta. \quad (14)$$

Thus Eq.(11) becomes

$$(b_m\alpha^2-\beta y_m)G_\alpha{}^m=-(2\beta^2+\alpha^2)\alpha^2 s_0/\beta. \quad (15)$$

Substituting Eq.(15) back into Eq.(12) yields

$$[(1+b^2)\alpha^4+(1+2b^2-g^2)+\beta^4]s_0=(b^2\alpha^2-\beta^2)\beta r_{00}/2. \quad (16)$$

Plugging Eq.(14) into Eq.(16) yields

$$[(1+b^2)\alpha^4+(1+2b^2-g^2)+\beta^4]\tau(x)=(b^2\alpha^2-\beta^2)r_{00}/2, \quad (17)$$

Suppose $(1+b^2)\alpha^4+(1+2b^2-g^2)+\beta^4$ is divisible by $b^2\alpha^2-\beta^2$, then we have

$$(1+b^2)\alpha^4+(1+2b^2-g^2)+\beta^4 =(b^2\alpha^2-\beta^2)(A\alpha^2+B\beta^2), \quad (18)$$

where A and B are assumed constant to be determined. In Eq.(18), comparison of the coefficients α^2, β^2 and $\alpha^2\beta^2$ yields

$$A=1+b^{-2},\quad B=-1. \quad (19)$$

and

$$g^2=2+3b^2+b^{-2}. \quad (20)$$

Since $g\in(-2,2)$, Eq.(20) is impossible. Thus $(1+b^2)\alpha^4+(1+2b^2-g^2)+\beta^4$ is not divisible by $b^2\alpha^2-\beta^2$. It follows from Eq.(17) that

$$\tau(x)=0. \quad (21)$$

By Eqs.(14), (15) and (17) we get

$$s_0=0, \quad (22)$$

$$(b_m\alpha^2-y_m\beta)G_\alpha^m=0, \quad (23)$$

$$r_{00}=0. \quad (24)$$

From Lemma 4 and Eq.(22), we know α is locally projectively flat. Substituting Eqs.(22)~(24) back into Eq.(9) yields

$$\alpha^2[g\beta+(1+b^2)\alpha](\beta+g\alpha)s_{l0}=0. \quad (25)$$

Note that $\alpha^2[g\beta+(1+b^2)\alpha](\beta+g\alpha)\neq 0$, thus we have

$$s_{l0}=0. \quad (26)$$

Because $s_{l0}=0$ and $r_{00}=0$, thus β is parallel with respect to α.

Conversely, if β is parallel with respect to α and α is locally projectively flat, by Lemma 1, we get $G^i=G_\alpha{}^i$. Because α is locally projectively flat, then F is locally projectively flat.

DOUGLAS TENSOR OF ASANOV FINSLER METRIC

Definition 1 Let

$$\begin{cases} D^i_{jkl}:=\dfrac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(G^i-\dfrac{1}{n+1}\dfrac{\partial G^m}{\partial y^m}y^i\right), \\ D:=D^i{}_{jkl}\mathrm{d}x^j\otimes\partial_i\otimes\mathrm{d}x^k\otimes\mathrm{d}x^l. \end{cases} \quad (27)$$

It is easy to verify that D is a well-defined tensor on TM_0. We say D is the Douglas tensor.

From Definition 1, it is easy to verify that the Douglas tensor is a non-Riemannian quantity, namely, if F is a Riemannian metric, then $D^i_{jkl}=0$. A Finsler metric is called a Douglas metric if $D^i_{jkl}=0$. The study on Douglas metrics will enhance our understanding on the geometric meaning of non-Riemannian quantities.

Lemma 5 $D^i_{jkl}=0$ if and only if

$$G^i-\frac{1}{n+1}\frac{\partial G^m}{\partial y^m}y^i=\gamma^i_{jk}(x)y^jy^k$$

for some set of local function $\gamma^i_{jk}(x)$.

It is known that the Douglas tensor is a projective invariant, namely, if two Finsler metrics F and $\bar{F}$ are projectively equivalent, i.e. $G^i=\bar{G}^i+Py^i$, where $P=P(x,y)$ is positively y-homogeneous of degree one, then the Douglas tensor of F is the same as that of $\bar{F}$. Thus, if a Finsler metric is projectively equivalent to a Berwald metric, then it is a Douglas metric. However, it is still an open problem whether or not every Douglas metric is (locally) projectively equivalent to a Berwald metric.

For Asanov Finsler metric Expression (2) by Lemma 1, denote

$$G^i:=G^i_\alpha+Py^i+Q^i. \quad (28)$$

Substituting Eq.(7) into Eq.(28) yields

$$G^i=G^i_\alpha+\frac{g}{g\beta+(1+b^2)\alpha}\left[-(g\alpha+\beta)s_0+\frac{1}{2}r_{00}\right]y^i+\frac{g}{g\beta+(1+b^2)\alpha}\left[-(g\alpha+\beta)s_0+\frac{1}{2}r_{00}\right]b^i+(g\alpha+\beta)s^i_0. \quad (29)$$

Compare Eq.(28) with Eq.(29), we get

$$P=\frac{g}{g\beta+(1+b^2)\alpha}\left[-(g\alpha+\beta)s_0+\frac{1}{2}r_{00}\right]y^i \quad (30)$$

$$Q^i=\frac{g}{g\beta+(1+b^2)\alpha}\left[-(g\alpha+\beta)s_0+\frac{1}{2}r_{00}\right]b^i+(g\alpha+\beta)s^i_0, \quad (31)$$

Let

$$\tilde{G}=y^i\frac{\partial}{\partial x^i}-2\tilde{G}^i\frac{\partial}{\partial y^i}, \quad (32)$$

then we get a new spray $\tilde{G}$, where

$$\tilde{G}^i:=G_\alpha{}^i+Q^i. \quad (33)$$

Clearly, G and $\tilde{G}$ are projectively equivalent. So we only need to compute the Douglas tensor of $\tilde{G}$. We can prove Theorem 2.

Proof (of Theorem 2) When $g=0$, F is a Riemannian metric. Theorem 2 naturally holds.

So in the following discussion we assume $g\neq 0$.

If the Douglas tensor of F vanishes, by Lemma 5, we have

$$\tilde{G}^i-\frac{1}{n+1}\frac{\partial\tilde{G}^m}{\partial y^m}y^i=\gamma^i{}_{jk}(x)y^jy^k. \quad (34)$$

Substituting Eq.(33) back into Eq.(27), we get

$$\tilde{D}^i_{jkl} := \bar{D}^i_{jkl} + \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(Q^i - \frac{1}{n+1}\frac{\partial Q^m}{\partial y^m} y^i\right),$$

where $\tilde{D}^i_{jkl}$ is the Douglas tensor of $\tilde{G}$, $\bar{D}^i_{jkl}$ is the Douglas tensor of α. Because α is a Riemannian metric, $\bar{D}^i_{jkl} = 0$. We get

$$Q^i - \frac{1}{n+1}\frac{\partial Q^m}{\partial y^m} y^i = \gamma^i_{jk}(x) y^j y^k. \tag{35}$$

Let $A=g\beta+(1+b^2)\alpha$. By Eq.(31), we get

$$\frac{\partial Q^m}{\partial y^m} = (1+b^2+g^2b^2)\alpha^2 s_0 / A^2 + \left[g(1+b^2)\alpha\beta - g^2\beta^2 - g\beta^3/\alpha\right]s_0 \big/ A^2 + g\left(\beta^2/\alpha - b^2\alpha\right) r_{00} \big/ (2A^2) + \frac{\alpha}{A} r_{m0} b^m. \tag{36}$$

Substituting Eq.(31) and Eq.(36) back into Eq.(34), we get

$$\begin{aligned}&(g\alpha+\beta)s^i_0 + \alpha b^i[-(g\alpha+\beta)s_0 + r_{00}/2]\cdot A^{-1} \\ &+ \frac{g(1+b^2)\alpha\beta - g^2\beta^2 - g\beta^3/\alpha}{(n+1)A^2} s_0 y^i \\ &- \frac{(1+b^2+g^2b^2)\alpha^2}{(n+1)A^2} s_0 y^i - \frac{g(\beta^2/\alpha - b^2\alpha)}{2(n+1)A^2} r_{00} y^i \\ &- \frac{\alpha}{(n+1)A} r_{m0} b^m y^i = \gamma^i_{jk}(x) y^j y^k.\end{aligned} \tag{37}$$

Let $B=(1+b^2)\alpha-g\beta$. Contracting Eq.(37) with b_i and replacing y by $-y$ yields

$$\begin{aligned}&(g\alpha-\beta)s_0 + b^2\alpha b^i[(g\alpha-\beta)s_0 - r_{00}/2]\cdot B^{-1} \\ &- \frac{[g(1+b^2)\alpha\beta - g^2\beta^2 + g\beta^3/\alpha]\beta}{(n+1)B^2} s_0 \\ &+ \frac{(1+b^2+g^2b^2)\alpha^2}{(n+1)B^2} s_0 - \frac{g(\beta^2/\alpha - b^2\alpha)\beta}{2(n+1)B^2} r_{00} \\ &+ \frac{\alpha\beta}{(n+1)B} r_{m0} b^m = -\gamma^i_{jk}(x) y^j y^k b_i.\end{aligned} \tag{38}$$

Eq.(37)+Eq.(38) yields

$$\begin{aligned}&2g\alpha s_0 - b^2\alpha^2[2(g^2-1-b^2)\beta s_0 - (1+b^2)r_{00}]\big/AB) \\ &-2g(1+b^2)(1+b^2+2g^2b^2)\alpha^4\beta^2 s_0\big/[(n+1)\alpha(AB)^2] \\ &-2g[(1+b^2)(1-3g^2)\alpha^2\beta^4 + g^2\beta^6]s_0\big/[(n+1)\alpha(AB)]^2 \\ &-g(\beta^2-b^2\alpha^2)\beta[(1+b^2)\alpha^2+g^2\beta^2]r_{00}\big/[(n+1)\alpha(AB)^2] \\ &+2g\alpha\beta^2 r_{m0}b^m\big/[(n+1)AB] = 0.\end{aligned} \tag{39}$$

Substituting $AB=(1+b^2)^2\alpha^2-g^2\beta^2$ into Eq.(39), we get

$$\begin{aligned}&[(1+b^2)^2\alpha^4 - b^2(g^2-1-b^2)\alpha^3\beta/g - 2g^2\alpha^2\beta^2]\alpha^2\beta s_0 \\ &+[gb^2(g^2-1-b^2)\alpha\beta^3 + g^3\beta^4]\alpha^2\beta s_0\big/(1+b^2)^2 \\ &-(1+b^2+2g^2b^2)\alpha^4\beta^2 s_0\big/(n+1) \\ &-\frac{1}{(n+1)}\left[(1-3g^2)\alpha^2\beta^2 + g^2\beta^4/(1+b^2)\right]\beta^2 s_0 \\ &-\frac{1}{(n+1)}\left[\left(1-\frac{g^2b^2}{1+b^2}\right)\alpha^2\beta^2 - b^2\alpha^4 + \frac{g^2\beta^2}{(1+b^2)^2}\right]r_{00} \\ &+\frac{1}{(n+1)}\left[\alpha^2 - g^2\beta^4/(1+b^2)^2\right]\alpha\beta^2 r_{m0}b^m = 0.\end{aligned} \tag{40}$$

Note Eq.(40) is a polynomial of y^i and α^{even} is also a polynomial of y^i, then α and α^2 must be zero. Since α^{odd} equals zero, we obtain

$$\begin{aligned}&b^2(g^2-1-b^2)[\alpha^2 - g^2\beta^2/(1+b^2)]\alpha^2 s_0 \\ &= [\alpha^2 - g^2\beta^2/(1+b^2)]r_{m0}b^m/(n+1).\end{aligned} \tag{41}$$

Since $\alpha^2-g^2\beta^2/(1+b^2)\neq 0$, then

$$b^2(g^2-1-b^2)\alpha^2 s_0 = r_{m0}b^m/(n+1), \tag{42}$$

Because $b^2(g^2-1-b^2)\alpha^2$ is not divisible by $r_{m0}b^m/(n+1)$, thus both sides of Eq.(42) are equal to zero. That is

$$s_0 = 0, \quad r_{m0}b^m = 0. \tag{43}$$

Substituting Eq.(43) back into Eq.(40), we obtain

$$-\frac{1}{n+1}\left[\left(1-\frac{g^2b^2}{1+b^2}\right)\alpha^2\beta^2 - b^2\alpha^4 + \frac{g^2\beta^4}{(1+b^2)^2}\right]r_{00} = 0.$$

Since

$$\frac{1}{n+1}\left[\left(1-\frac{g^2b^2}{1+b^2}\right)\alpha^2\beta^2 - b^2\alpha^4 + \frac{g^2\beta^4}{(1+b^2)^2}\right] \neq 0,$$

we have

$$r_{00} = 0. \tag{44}$$

By Eqs.(43) and (44), we get $b_{i|j}=0$, which shows β is parallel with respect to α.

Conversely, if β is parallel with respect to α, by Lemma 1, we get $G^i=G_\alpha^{\ i}$. Because α is a Riemannian metric, $D^i{}_{jkl}=0$.

Lemma 6 (Shen, 2004) For an (α,β)-metric $F=\alpha\phi(\beta/\alpha)$ on a manifold of dimension $n\geq 3$, the following are equivalent:

(1) F is a Landsberg metric;

(2) F is a Berwald metric;

(3) β is parallel with respect to α.

By Theorem 2 and Lemma 6, we have

Corollary If the Asanov Finsler metric $F=\alpha(\beta^2/\alpha^2+g\beta/\alpha+1)^{1/2}\exp\{(G/2)\arctan[\beta/(h\alpha)+G/2]\}$ is a Douglas metric, then

(1) F is a Berwald metric;

(2) F is a Landsberg metric.

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