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Research on 3D fiber orientation distribution in arbitrary planar flows*

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Abstract: A non-stretchable fiber rotation in planar flows has been solved. The fiber will rotate periodically or run to the asymptotical direction decided by a discriminant defined in the paper involving the fiber aspect ratio and the flow characteristics. Subsequently the fiber orientation distribution is derived directly without the bother of solving the Fokker-Planck equation. The research clearly indicates the overall configuration of a fiber rotation movement in planar flows.

Key words: Fiber suspension flow, Jeffery orbit, Fiber orientation distribution
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INTRODUCTION

Fiber suspension flow has great significance both on scientific research and industrial applications. A fiber is a slender body and is always modelled as a rigid cylinder with length l and diameter d. There are two main aspects in fiber suspension flow research. One aspect is to study the fiber movement in flows. The other aspect is to investigate the effect of fiber additives on the flow.

Jeffery (1922) first studied the motion of an ellipsoid immersed in simple shear Newtonian flow with neglecting the inertia and Brownian rotation. He found that an ellipsoid rotates periodically and that the trace of one end of the ellipsoid was characterized by the known Jeffery orbits. Bretherton (1962) showed that the same equations could be used to describe the motion of any axisymmetric particle provided that one used an equivalent aspect ratio $r_e=l/d$ that is equal to the actual aspect ratio r_p for ellipsoidal particles, and that for cylindrical particle $r_e\approx 0.7r_p$. According to (Jeffery, 1922), a fiber rotation equation is:

$$\dot{\boldsymbol{p}} = \boldsymbol{\omega}\cdot\boldsymbol{p} + \lambda(\boldsymbol{\varepsilon}\cdot\boldsymbol{p} - \boldsymbol{\varepsilon}:\boldsymbol{ppp}), \tag{1}$$

where $\boldsymbol{p}$ is the unit vector aligned with the fiber axis, $\boldsymbol{\omega}=(\nabla\boldsymbol{u}^{\mathrm{T}}-\nabla\boldsymbol{u})/2$ is the vorticity tensor, $\boldsymbol{\varepsilon}=(\nabla\boldsymbol{u}^{\mathrm{T}}+\nabla\boldsymbol{u})/2$ is the deformation rate tensor, $\lambda=(r_c^2-1)/(r_c^2+1)$.

For the second main problem, Batchelor (1970a) developed a general constitutive equation for suspensions of particles of any shape in Newtonian liquids at arbitrary concentrations, which gave the relationship between the microstructure of particles and the macroscopic property of the solution. As the model is too complicated to use, Batchelor (1970b; 1971) simplified the model for slender fibers in a dilute suspension

$$\boldsymbol{\tau} = 2\mu\boldsymbol{\varepsilon} + \mu_{\mathrm{f}}\left(\langle\boldsymbol{pppp}\rangle - \frac{1}{3}\boldsymbol{I}\langle\boldsymbol{pp}\rangle\right):\boldsymbol{\varepsilon}, \tag{2}$$

$$\boldsymbol{A}_4 = \langle\boldsymbol{pppp}\rangle = \oint\boldsymbol{pppp}\psi\,\mathrm{d}\boldsymbol{p}, \tag{3}$$

$$\boldsymbol{A}_2 = \langle\boldsymbol{pp}\rangle = \oint\boldsymbol{pp}\psi\,\mathrm{d}\boldsymbol{p}, \tag{4}$$

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where, $\boldsymbol{\tau}$ is the deviatoric stress of suspension flow, μ is the Newtonian fluid viscosity, μ_f is the apparent viscosity brought by fiber suspension, ψ is the orientation distribution function.

The orientation distribution function ψ in the Lagrangian representation satisfies the Fokker-Planck equation

$$\frac{\partial \psi}{\partial t} + \nabla_p \cdot (\psi \dot{\boldsymbol{p}}) = 0, \tag{5}$$

which means that the distribution function is conservative in the fiber orientation probability space, with the gradient ∇_p being computed over the fiber configuration space.

Recently, some researchers (Lin *et al*., 2002; Zhang and Lin, 2003; Zhang *et al*., 2005; Zhou and Lin, 2005) developed a series of research of 2D fiber suspension under different flow conditions. In the paper 3D fiber orientation distribution has been investigated, which is a much better approximation of the practice. The analytical solution of the fiber evolution Eq.(1) in planar flows is presented. Double Lagrange representation method (Szeri and Leal, 1992) and the foregoing analytical solution were used to obtain the 3D fiber orientation distribution. The periodicity, asymptotical direction, etc. were studied thoroughly.

FIBER ROTATION ORBIT

In planar flows, we denote $\partial u/\partial y = \dot{\gamma}$, and $\partial v/\partial x = k\dot{\gamma}$, $\partial u/\partial x = j\dot{\gamma}$, $\partial v/\partial y = -j\dot{\gamma}$, in which the incompressible condition has been taken into account. Since the fiber is not stretchable, it is convenient to depict fiber orientation in the spherical coordinates $p_x=\sin\theta\cos\phi$, $p_y=\sin\theta\sin\phi$, $p_z=\cos\theta$, then Eq.(1) is transformed to:

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = [\lambda(k+1)\cos(2\phi) + k - 1 - 2\lambda j \sin(2\phi)]\dot{\gamma}/2, \tag{6}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \lambda \sin(2\theta)[2j\cos(2\phi) + (k+1)\sin(2\phi)]\dot{\gamma}/4. \tag{7}$$

Note that in Eq.(6), ϕ only explicitly depends on t, so we can separate the variables and integrate:

$$\int_{\phi_0}^{\phi} [\lambda(k+1)\cos(2\xi) + k - 1 - 2\lambda j \sin(2\xi)]^{-1} \mathrm{d}\xi = \frac{1}{2}\int_0^t \dot{\gamma}\mathrm{d}t, \tag{8}$$

then

$$\phi = \arctan\left\{\left\{\tanh\left[\operatorname{arctanh}\left(\frac{(\lambda k + \lambda - k + 1)\tan\phi_0 + 2\lambda j}{\sqrt{\Delta}}\right) + \frac{1}{2}\dot{\gamma}t\sqrt{\Delta}\right]\sqrt{\Delta} - 2\lambda j\right\}(\lambda k + \lambda - k + 1)^{-1}\right\}, \tag{9}$$

where

$$\Delta = \lambda^2 k^2 + 2\lambda^2 k + \lambda^2 - k^2 + 2k - 1 + 4\lambda^2 j^2. \tag{10}$$

For convenience, in Eq.(7) we denote

$$f(\phi(t)) = [2j\cos(2\phi) + (k+1)\sin(2\phi)]\dot{\gamma}/4,$$

then by separating the variables and integrating we have

$$\theta = \arctan\left(\exp\left(\int_0^t f(\phi(\xi))\mathrm{d}\xi\right)\tan\theta_0\right). \tag{11}$$

And

$$\int f(\phi(t))\mathrm{d}t = \int f(\phi)\left(\frac{\mathrm{d}t}{\mathrm{d}\phi}\right)\mathrm{d}\phi = \frac{1}{2}\ln\frac{1+\tan^2\phi}{(\lambda k + \lambda - k + 1)(\tan^2\phi + 4\lambda j \tan\phi - \lambda k - \lambda - k + 1)}, \tag{12}$$

so substituting Eq.(12) into Eq.(11) gives

$$\theta = \arctan\left(\frac{\sqrt{2\lambda j \sin(2\phi_0) - \lambda(k+1)\cos(2\phi_0) - k + 1}\,\sin\theta_0}{\sqrt{2\lambda j \sin(2\phi) - \lambda(k+1)\cos(2\phi) - k + 1}\,\cos\theta_0}\right). \tag{13}$$

From Eqs.(9) and (13) it is shown that the fiber rotational periodicity is decided by ϕ evolution, since ϕ is independent of θ. In Eq.(9) arctangent is monotonic, then the periodicity of the compound function is decided by arc hyperbolic tangent. If only the domain is imaginary, arc hyperbolic tangent is periodic. As the fiber rotation period relative to ϕ is 2π, then the time period can be determined

$$T = 4\pi/(\dot{\gamma}\sqrt{-\Delta}). \tag{14}$$

If only Δ is negative a periodical solution occurs, otherwise an asymptotical solution appears.

Let $d\phi/dt=0$ in Eq.(6), then

$$\phi_{\mathrm{d}} = \mathrm{Re}\left(\arctan\frac{-2\lambda j \pm \Delta}{\lambda(k+1)+1-k}\right), \tag{15}$$

where ϕ_{d} is the extremum direction or asymptotical direction respectively for periodical movement or for non-periodical movement.

Fig.1 represents the famous Jeffery orbits for different aspect ratios. A fiber will rotate periodically in simple shear flow, with the orbit being totally decided by the initial orientation, independent of the shear rate. All orbits gather in the flow direction, which means a high orientation distribution in the direction. Comparing Figs.1a and 1b, it is obvious that fibers increase their alignment with increasing aspect ratio. In fact, the aspect ratio effect is similar in all kinds of flows, except that the increasing of alignment is in the direction decided by Eq.(15) according to different flow conditions. Fig.2 shows the orbits in double shear flow. Figs.2a and 2b have the same shearing magnitude, but absolutely different results because of the difference of two shearing directions. Since in Fig.2a the discriminant, i.e., Eq.(15), is negative, the fiber will rotate periodically. Specially, when two shearing magnitudes are equal, the orbits become a group of circles. Fig.2a shows the asymptotical orbits; all fibers will finally orientate towards the only direction. Eq.(15) gives two directions, one is the stable asymptotical direction and the other unstable. Besides, the two directions are not perpendicular to one another, so the orbits are not mirror-symmetric any more. Similar results are for the shear-extension combined flows (Fig.3).

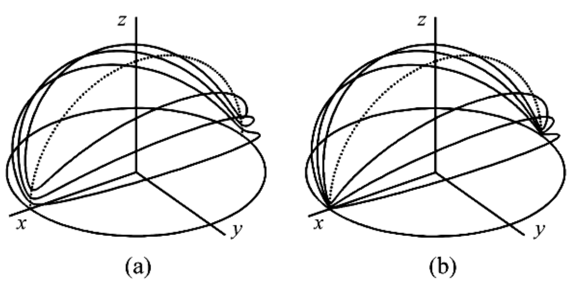

Fig.1 Jeffery orbits of different orbit constants
(a) λ=99/101; (b) λ=999/1001

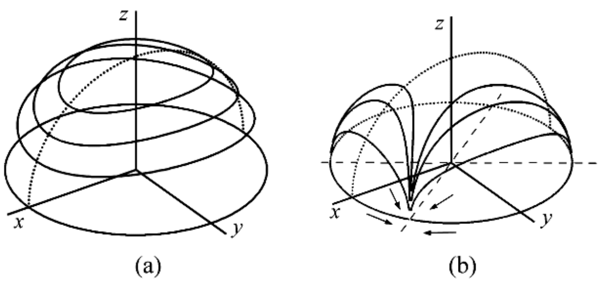

Fig.2 Fiber orientation orbits in double shear flow
(a) k=−0.5; (b) k=0.5

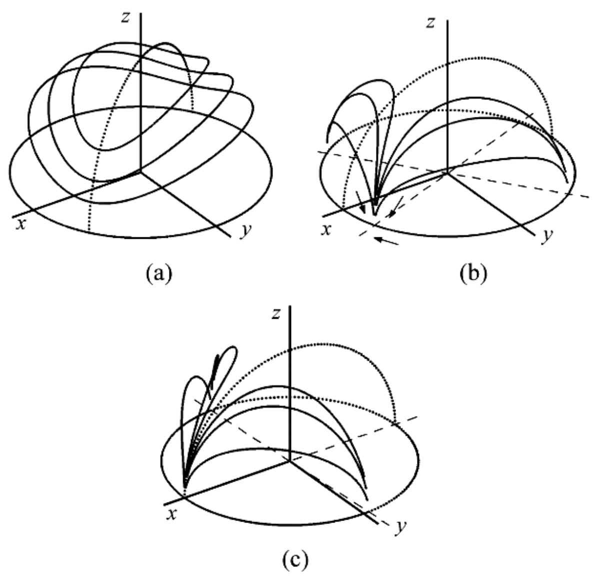

Fig.3 Fiber orientation orbits in shear and extension combined flow. (a) k=−0.5, j=−0.5; (b) k=0.5, j=0.5; (c) j=10. The dash lines in (b) and (c) denote the approximation directions. Orbits in (c) is the approximation of almost-extension flow case

In the foregoing process, the simple extension flow case is omitted, since it cannot be integrated to the discussion. Separate treatment is trivial and will give not much extra information. It is natural and effective to approximate the simple extension flow using large extension over shearing ratio, j. A very good approximate result is shown in Fig.3c, which indicates that fibers will asymptotically align in the extensional direction.

FIBER ORIENTATION DISTRIBUTION

In Lagrangian coordinates moving along with the flow, the conservation of fiber orientation distribution is depicted by the Fokker-Planck Eq.(5). Szeri and Leal (1992) further adopted the Lagrangian representation for the distribution along the orbits and developed a so-called double Lagrangian representation method to solve the Fokker-Planck equation. The basic idea of the method is briefly introduced below.

First, the distribution function is confined to arbitrary one of the fiber orbits, i.e.

$$\psi^*(t;\boldsymbol{p}_0)=\psi(t,\boldsymbol{p})|_{p=\hat{p}(t;p_0)}, \tag{16}$$

where $\hat{\boldsymbol{p}}(t;\boldsymbol{p}_0)$ is the solution of the associated fiber orientation evolution with initial condition $\boldsymbol{p}_0$. It means a Lagrangian representation moving along with the selected orbit, in addition with the Lagrangian representation in the Fokker-Planck equation where the nomenclature double Lagrangian representation method (DLR) comes from. The time derivative of ψ^* can be computed from the definition Eq.(16) and from the ψ evolution Eq.(5):

$$\frac{\partial}{\partial t}\psi^*(t;\boldsymbol{p}_0)=\left[\frac{\partial\psi}{\partial t}+\nabla\psi\cdot\dot{\boldsymbol{p}}\right]\Bigg|_{p=\hat{p}(t;p_0)}$$

$$=[-\psi\nabla\cdot\dot{\boldsymbol{p}}]\big|_{p=\hat{p}(t;p_0)}=-\psi^*(t;\boldsymbol{p}_0)[\nabla\cdot\dot{\boldsymbol{p}}]\big|_{p=\hat{p}(t;p_0)}. \tag{17}$$

With some variables transformation and calculation (for details please refer to the original), it gives

$$\frac{\psi^*(t;\boldsymbol{p}_0)}{\psi^*(0;\boldsymbol{p}_0)}=\frac{1}{\det(\nabla_0\hat{\boldsymbol{p}}(t;\boldsymbol{p}_0))}. \tag{18}$$

In unit spherical surface, the gradient operator is

$$\nabla_0=\boldsymbol{e}_\phi\frac{1}{\sin\theta_0}\frac{\partial}{\partial\phi_0}+\boldsymbol{e}_\theta\frac{\partial}{\partial\theta_0}, \tag{19}$$

so Eq.(18) is simplified as

$$\frac{\psi^*(t;\phi_0,\theta_0)}{\psi^*(0;\phi_0,\theta_0)}=\frac{\sin\theta_0}{\sin\theta}\left(\frac{\partial\phi}{\partial\phi_0}\frac{\partial\theta}{\partial\theta_0}-\frac{\partial\theta}{\partial\phi_0}\frac{\partial\phi}{\partial\theta_0}\right)^{-1}. \tag{20}$$

Eq.(20) gives the orientation distribution evolution in a very simple way. The fiber orientation distribution is totally decided by the fiber rotation orbit and the initial orientation. Although Szeri and Leal (1992) presented the DLR as a novel method, yet there is universal theory to deal with first order partial differential equation (Courant and Hilbert, 1966). Using the characteristic method, a first order partial differential equation can be converted into a group of ordinary differential equations if only the characteristics exist. The DLR is a characteristic method in nature. After the fiber rotation orbit has been obtained in the above section, then after substituting the result into Eq.(20), the orientation distribution can be obtained explicitly.

An initial distribution is needed to compute the distribution evolution. We will take the mean initial distribution condition, which means that the fibers are initially isotropically oriented. Since the Double Lagrangian representation method is to evaluate the distribution along the fiber orbit, different distribution results separately corresponding to the orbits in Figs.1~3 are given in Figs.4~6. In simple shear flow, the orientation distribution peaks in the flow direction, and larger aspect ratio increases the aligning effect greatly (Figs.4a, 4b). The maximum value of distribution appears in the direction decided by Eq.(15). The distribution is much higher when the fiber is lying in the flow plane than out of the plane, which means most fibers are running in the flow plane.

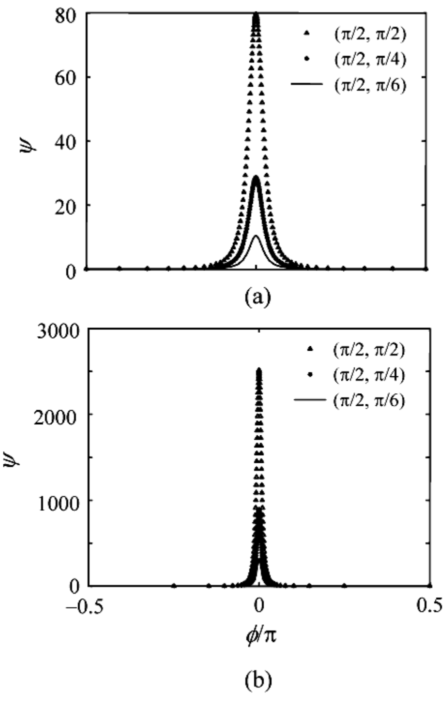

Fig.4 Orientation distribution along different Jeffery orbits. (a) λ=99/101; (b) λ=999/1001

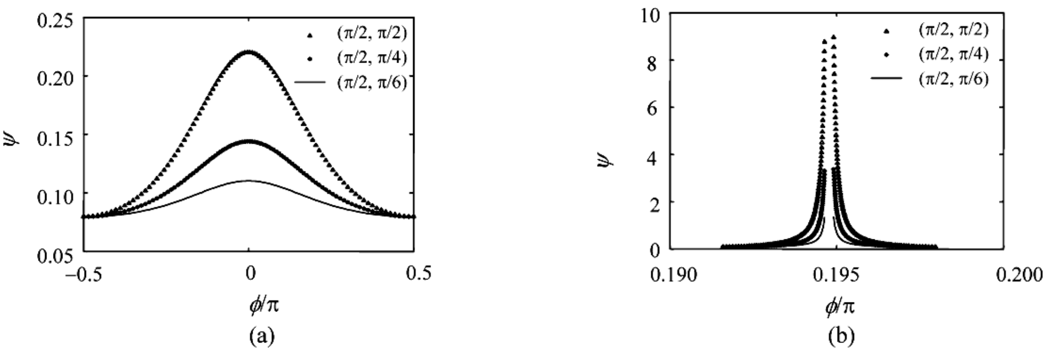

Fig.5 Orientation distribution in double shear flow. (a) k=−0.5; (b) k=0.5

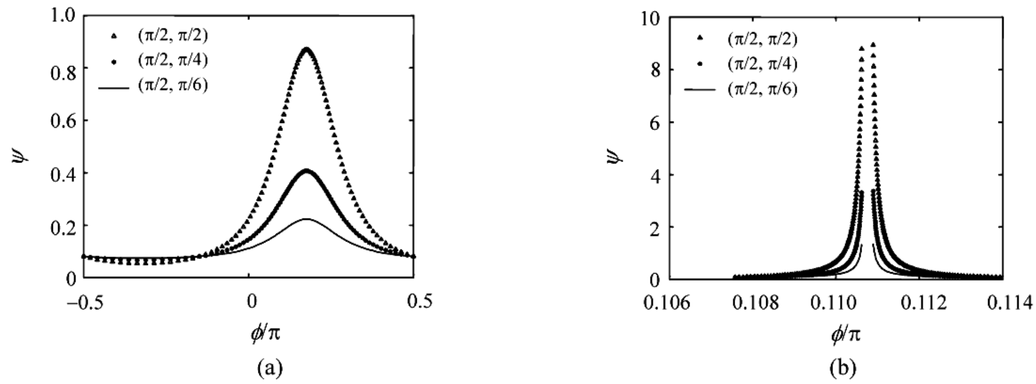

Fig.6 Orientation distribution in planar flow. (a) k=−0.5, j=−0.5; (b) k=0.5, j=0.5

CONCLUSION

A 3D fiber rotation in arbitrary planar has been completely solved. The well-known Jeffery orbit in simple shear flow is extended to the case of arbitrary planar flow. It is found that a fiber may rotate periodically or approach to an asymptotic direction, which is determined by a new found rule. The period or asymptotic direction has been clearly formulated. Simultaneously, the fiber orientation distribution has been derived by a novel method, avoiding the complicate solving process of the Fokker-Planck equation.

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