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Constrained multi-degree reduction of rational Bézier curves using reparameterization*

CAI Hong-jie, WANG Guo-jin‡

(*Institute of Computer Images and Graphics, State Key Laboratory of CAD & CG, Zhejiang University, Hangzhou 310027, China*)

E-mail: cai_hongjie@sina.com; wanggj@zju.edu.cn

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Abstract: Applying homogeneous coordinates, we extend a newly appeared algorithm of best constrained multi-degree reduction for polynomial Bézier curves to the algorithms of constrained multi-degree reduction for rational Bézier curves. The idea is introducing two criteria, variance criterion and ratio criterion, for reparameterization of rational Bézier curves, which are used to make uniform the weights of the rational Bézier curves as accordant as possible, and then do multi-degree reduction for each component in homogeneous coordinates. Compared with the two traditional algorithms of "cancelling the best linear common divisor" and "shifted Chebyshev polynomial", the two new algorithms presented here using reparameterization have advantages of simplicity and fast computing, being able to preserve high degrees continuity at the end points of the curves, do multi-degree reduction at one time, and have good approximating effect.

Key words: Rational Bézier curves, Constrained multi-degree reduction, Reparameterization

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INTRODUCTION

For computer aided geometric design, the highest degree of parametric curves or surfaces that different design systems can code with may be different, and for the convenience of data transmission and data exchange between those systems, approximate degree reduction for the curves and surfaces are needed. What is more, degree reduction can decrease the information storage. So, study of degree reduction algorithms has many practical uses.

Research on degree reduction of polynomial curves in L_2 or L_∞ norm has achieved much progress (Hu *et al*., 1998; Zhang and Wang, 2005; Lu and Wang, 2006a; 2006b), whereas so far for rational curves, there exist only two algorithms, i.e., cancelling the best linear common divisor (Sederberg and Chang, 1993) and shifted Chebyshev polynomial (Chen, 1994) as main production. The former used Chebyshev polynomial, and the latter can preserve the end points of the reduced curves unchanged. These two algorithms are both remarkable in theory, but cannot do multi-degree reduction at one time; also their errors are related to the magnitude of the denominator of the original curves. Chen and Wang (2000) gave an approximate error evaluation for the algorithms in (Chen, 1994; Sederberg and Chang, 1993). However, as the evaluation requires approximations of inequalities, the final result of the evaluation deviated greatly from the precise error. Park and Lee (2005) induced degree reduction algorithm for polynomial curves to one for rational case, in which there is also no multi-degree reduction at one time. Based on the analyses above, we know that good degree reduction algorithms for rational Bézier curves are insufficient whereas their practical uses

‡ Corresponding author

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are quite large. Therefore, to diminish the gap, research on new and effective degree reduction algorithms, especially on optimal constrained multi-degree reduction algorithms for rational Bézier curves, are urgently needed.

Thus, observing that Zhang and Wang (2005) is the latest work that has perfect result on constrained multi-degree reduction for polynomial Bézier curves, we extend it to one for rational Bézier curves using homogeneous coordinates. The basic idea is introducing variance criterion and ratio criterion to reparameterize original rational curve such that its weights can be made uniform to the utmost, and then do constrained multi-degree reduction for each component in homogeneous coordinates. The method can not only perform a constrained multi-degree reduction for the original curves, but also minimize the error. Many numerical experiments showed that compared with "cancelling the best linear common divisor" (Sederberg and Chang, 1993) and "shifted Chebyshev polynomial" (Chen, 1994), the new algorithms have advantages to compute simply and rapidly, to preserve high degree continuity at the end points of the curves, to do multi-degree reduction for the original curves at one time, and to approximate reduced curves with high precision.

CONSTRAINED MULTI-DEGREE REDUCTION FOR RATIONAL BÉZIER CURVES USING HOMOGENEOUS COORDINATES

For convenience, without loss of generality, we only discuss the degree reduction of planar rational curves.

In affine coordinates, a planar rational Bézier curve of degree n can be presented as

$$\boldsymbol{R}(t)=(x(t),y(t))=\sum_{i=0}^{n}B_i^n(t)\omega_i\boldsymbol{R}_i\Big/\sum_{i=0}^{n}B_i^n(t)\omega_i,\ 0\le t\le 1, \tag{1}$$

where $B_i^n(t)$ are Bernstein bases, $\boldsymbol{R}_i=(x_i,y_i)$ are control points and ω_i are weights.

The same curve can be presented in homogeneous coordinates as

$$\tilde{\boldsymbol{R}}(t)=(X(t),Y(t),\omega(t))=\sum_{i=0}^{n}B_i^n(t)\tilde{\boldsymbol{R}}_i,\ 0\le t\le 1, \tag{2}$$

where $\tilde{\boldsymbol{R}}_i=(X_i,Y_i,\omega_i)=(\omega_i x_i,\omega_i y_i,\omega_i)$ are control points.

Theorem 1 An n-degree planar rational Bézier curve (2) can be multi-degree reduced to an m-degree ($m<n$) one, which preserves ($v-1$)- and ($u-1$)-degree continuity ($v-1\ge 0$, $u-1\ge 0$ and $u+v\le m+2$) at the end points and can be presented in homogeneous coordinates as follows:

$$\bar{\boldsymbol{R}}(t)=(\bar{X}(t),\bar{Y}(t),\bar{\omega}(t))=\sum_{i=0}^{m}B_i^m(t)\bar{\boldsymbol{R}}_i,$$
$$\bar{\boldsymbol{R}}_i=(\bar{X}_i,\bar{Y}_i,\bar{\omega}_i),\ 0\le t\le 1,$$

where the homogeneous coordinates of the control points of the degree reduced curve can be presented as:

$$\begin{pmatrix}\bar{X}_0 & \bar{Y}_0 & \bar{\omega}_0\\ \bar{X}_1 & \bar{Y}_1 & \bar{\omega}_1\\ \cdots & \cdots & \cdots\\ \bar{X}_m & \bar{Y}_m & \bar{\omega}_m\end{pmatrix}=\boldsymbol{G}_{(m+1)\times(n+1)}^{n,m,u,v}\cdot\begin{pmatrix}\omega_0x_0 & \omega_0y_0 & \omega_0\\ \omega_1x_1 & \omega_1y_1 & \omega_1\\ \cdots & \cdots & \cdots\\ \omega_nx_n & \omega_ny_n & \omega_n\end{pmatrix},$$

where

$$\boldsymbol{G}_{(m+1)\times(n+1)}^{n,m,u,v}=\boldsymbol{F}_{(m+1)\times(m+1)}^{m}\boldsymbol{B}_{(m+1)\times(n+1)}^{n,m}-\boldsymbol{F}_{(m+1)\times(m-v+1)}^{m,v}\cdot\boldsymbol{A}_{(m-v+1)\times(n-m)}^{n,m,u,v}\left(\boldsymbol{A}_{(n-m)\times(n-m)}^{n,m,u,v}\right)^{-1}\boldsymbol{B}_{(n-m)\times(n+1)}^{n,m},$$

and

$$\boldsymbol{A}_{(n-v+1)\times(n-m)}^{n,m,u,v}=\begin{pmatrix}\boldsymbol{A}_{(m-v+1)\times(n-m)}^{n,m,u,v}\\ \boldsymbol{A}_{(n-m)\times(n-m)}^{n,m,u,v}\end{pmatrix}$$
$$=\begin{pmatrix}c_0^{m-u-v+1} & c_0^{m-u-v+2} & \cdots & c_0^{n-u-v-1} & c_0^{n-u-v}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ c_{m-v+1}^{m-u-v+1} & c_{m-v+1}^{m-u-v+2} & \cdots & c_{m-v+1}^{n-u-v-1} & c_{m-v+1}^{n-u-v}\\ 0 & c_{m-v+2}^{m-u-v+2} & \cdots & c_{m-v+2}^{n-u-v-1} & c_{m-v+2}^{n-u-v}\\ 0 & 0 & \cdots & c_{m-v+3}^{n-u-v-1} & c_{m-v+3}^{n-u-v}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & c_{n-v}^{n-u-v}\end{pmatrix},$$

$$c_j^l=\begin{cases}\displaystyle\sum_{k=0}^{l}(-1)^{k-j}\binom{l+2u}{l-k}\binom{l+2u+2v+k}{k}\binom{k+u}{j}, & 0\le j\le u,\\ \displaystyle\sum_{k=j-u}^{l}(-1)^{k-j}\binom{l+2u}{l-k}\binom{l+2u+2v+k}{k}\binom{k+u}{j}, & u+1\le j\le l+u,\end{cases}$$
$$l=m-u-v+1,m-u-v+2,\ldots,n-u-v,$$

$$\boldsymbol{B}^{n}_{(n+1)(n+1)}=\begin{pmatrix}\boldsymbol{B}^{n,m}_{(m+1)\times(n+1)}\\ \boldsymbol{B}^{n,m}_{(n-m)\times(n+1)}\end{pmatrix}=(b_{i,j})_{(n+1)(n+1)},$$

$$b_{i,j}=\begin{cases}0, & i<j,\\ (-1)^{i+j}\binom{n}{i}\binom{i}{j}, & i\geq j,\end{cases}\quad i,j=0,1,\ldots,n,$$

$$\boldsymbol{F}^{m}_{(m+1)\times(m+1)}=\left(\boldsymbol{F}^{m,v}_{(m+1)\times v},\boldsymbol{F}^{m,v}_{(m+1)\times(m-v+1)}\right)=(f_{i,j})_{(m+1)\times(m+1)},$$

$$f_{i,j}=\begin{cases}0, & i<j,\\ \binom{m-j}{i-j}\Big/\binom{m}{i}, & i\geq j,\end{cases}\quad i,j=0,1,...,m.$$

Proof According to the degree reduced theorem in (Zhang and Wang, 2005), it is easy to know that the curve $\overline{\boldsymbol{R}}(t)$'s three homogeneous coordinates ($\overline{X}(t)$, $\overline{Y}(t),\overline{\omega}(t)$) and the original curve $\tilde{\boldsymbol{R}}(t)$'s counterpart ($X(t)$, $Y(t)$, $\omega(t)$) have the same derivatives up to degree $v-1$ (≥ 0) and $u-1$ (≥ 0) at the end points of [0, 1], respectively; also the former is the best ($n-m$)-degree reduction polynomial of the latter in L_2-norm. Then applying the law of four arithmetic operations of differential, Theorem 1 is proved.

According to (Zhang and Wang, 2005), the three homogeneous coordinates of the degree reduced curve $\overline{\boldsymbol{R}}(t)$ are the best constrained multi-degree reduction for those of the original curve $\tilde{\boldsymbol{R}}(t)$ respectively, and the degree reduced error has a direct presentation which can estimate whether it is smaller than the tolerance ε^*.

Algorithm 1 Applying the matrix formulas in Theorem 1, given an n-degree rational Bézier curve $\boldsymbol{R}(t)$, we can achieve an ($n-m$)-degree ($0<m<n$) reduced curve $\overline{\boldsymbol{R}}(t)$ which preserves ($v-1$)- and ($u-1$)-degree continuity ($v-1\geq 0$, $u-1\geq 0$) at the end points.

CONSTRAINED MULTI-DEGREE REDUCTION FOR RATIONAL BÉZIER CURVES USING REPARAMETERIZATION

It is well known that rational Bézier curves can be formally treated as polynomial Bézier curves by using homogeneous coordinates, but these two are different in nature unless all weights of the rational Bézier curves are equal. That means Algorithm 1 cannot obtain the minimal degree reduction error generally. However we can make uniform the weights of the rational Bézier curves using reparameterization to diminish the degree reduction error. So, we should reparameterize the rational Bézier curves before applying Algorithm 1, to make uniform the weights as accordant as possible. Based on the discussion above, we will give a new algorithm using the variance criterion and then try to introduce another criterion called "ratio criterion" (Zheng, 2005) for reparameterization to implement the second algorithm.

Reparameterization by variance criterion

By introducing the parametric transformation $\hat{t}=ct/[1+(c-1)t]$, the control points and shape of the rational curve (1) remain unchanged and the weights are changed to $\hat{\omega}_i=c^{n-i}\omega_i$, where c is the reparameterization factor, a positive real number. To determine c, we will select a criterion for measuring the uniformity level of the weights. Certainly, a natural and direct idea is to select the variance of all weights $\sum_{i=0}^{n}(\omega_i-\overline{\omega})^2$ as a criterion, where $\overline{\omega}=\sum_{i=0}^{n}\omega_i\big/(n+1)$. But further analysis reveals that the weights do not have absolute values but only relative values, so variance criterion can be improved to $g=\sum_{i=0}^{n}\left(\omega_i/\overline{\omega}-1\right)^2$, or expand the expression of g to get $g=(n+1)^2h-(n+1)$, where $h=\left(\sum_{i=0}^{n}\omega_i^2\right)\cdot\left(\sum_{i=0}^{n}\omega_i\right)^{-2}$.

It is obvious that g and h have linear correlation and that the smaller g becomes, the smaller h also becomes. Thus we can choose h instead of g as the criterion for measuring the accordance degree of the weights.

Provided the measuring criterion, we will proceed to determine the factor c so that the value of h can be taken as minimal when the above mentioned parametric transformation to the rational curve (1) has been implemented. As new weights become $\hat{\omega}_i=c^{n-1}\omega_i$ after the parametric transformation, the new value of h can be written as

$$h(c)=\sum_{i=0}^{n}(c^{n-i}\omega_i)^2\Bigg/\left(\sum_{i=0}^{n}c^{n-i}\omega_i\right)^2. \qquad (3)$$

Obviously, what we need to do is to find all zero points of the expression

$$\left(\sum_{i=0}^{n}c^{n-i}\omega_i\right)^4\cdot\frac{\mathrm{d}h(c)}{\mathrm{d}c}=\frac{\mathrm{d}}{\mathrm{d}c}\left(\sum_{i=0}^{n}(c^{n-i}\omega_i)^2\right)\left(\sum_{i=0}^{n}c^{n-i}\omega_i\right)^2 -2\sum_{i=0}^{n}(c^{n-i}\omega_i)^2\sum_{i=0}^{n}c^{n-i}\omega_i\frac{\mathrm{d}}{\mathrm{d}c}\left(\sum_{i=0}^{n}c^{n-i}\omega_i\right), \quad (4)$$

and to select a positive one from them such that Eq.(3) has the minimal value on it, and that is the factor c we want. Mathematical software such as Matlab can do these procedures. And the existence of the positive real number c can be assured by the following theorem.

Theorem 2 The expression $h(c)$ written as Eq.(3) reaches the minimal value in $(0,+\infty)$.

Proof For an arbitrary negative number c, we have $h(c)>h(-c)$. That means, by replacing c with a positive number $-c$, the value of h decreases. So it is impossible for $h(c)$ to reach the minimum in $(-\infty, 0)$. A few computations give $h(0)=1$, $h(1)<1$, $\lim_{c\to+\infty}h(c)=1$. Since $h(c)$ is continuous, according to mathematical analysis, it reaches its minimum in $(0,+\infty)$.

Combining the idea of variance criterion for reparameterization with Theorem 2 and Algorithm 1, we have

Algorithm 2 First perform reparameterization by variance criterion for the rational Bézier curve (2), and secondly do constrained multi-degree reduction to the curve in homogeneous coordinates. Then each component in homogeneous coordinates of the control points of the degree reduced curve can be given by

$$\begin{pmatrix}\bar{X}_0 & \bar{Y}_0 & \bar{\omega}_0\\ \bar{X}_1 & \bar{Y}_1 & \bar{\omega}_1\\ \cdots & \cdots & \cdots\\ \bar{X}_m & \bar{Y}_m & \bar{\omega}_m\end{pmatrix}=\boldsymbol{G}^{n,m,u,v}_{(m+1)\times(n+1)}\cdot\begin{pmatrix}\hat{\omega}_0x_0 & \hat{\omega}_0y_0 & \hat{\omega}_0\\ \hat{\omega}_1x_1 & \hat{\omega}_1y_1 & \hat{\omega}_1\\ \cdots & \cdots & \cdots\\ \hat{\omega}_nx_n & \hat{\omega}_ny_n & \hat{\omega}_n\end{pmatrix}, \quad (5)$$

where $\hat{\omega}_i=c^{n-i}\omega_i\ (i=1,2,\cdots,n)$ and c is obtained by selecting the positive number from all zero points of Eq.(4) using mathematical software, on which Eq.(3) takes its minimal value.

Reparameterization by ratio criterion

Different from the variance criterion, Zheng (2005) proposed another criterion called "ratio criterion" for reparameterization to make uniform the weights of the rational Bézier curves. The ratio criterion is given as

$$\rho=\max_{0\le i\le n}\{\gamma^i\omega_i\}\big/\min_{0\le i\le n}\{\gamma^i\omega_i\} =\max_{0\le i,j\le n}\{\gamma^{i-j}\omega_i/\omega_j\}.$$

The minimal point and the minimum of the expression above is

$$\gamma=\exp\left(-\frac{\underline{W}_{j_0}+\overline{W}_{k_0}}{j_0+k_0}\right),\ \rho=\exp\left(\frac{j_0\overline{W}_{k_0}-k_0\underline{W}_{j_0}}{j_0+k_0}\right), \quad (6)$$

respectively [we found an error in (Zheng, 2005), that the expression of H_0 is wrong, and Eq.(6) here is the correct one], where the integrals j_0 and k_0 satisfy

$$\frac{j_0\overline{W}_{k_0}-k_0\underline{W}_{j_0}}{j_0+k_0}=\max_{1\le j,k\le n}\frac{j\overline{W}_k-k\underline{W}_j}{j+k}, \quad (7)$$

and

$$\overline{W}_k=\max_{k\le i\le n}\{\log\omega_i-\log\omega_{i-k}\},$$
$$\underline{W}_k=\min_{k\le i\le n}\{\log\omega_i-\log\omega_{i-k}\},\ k=1, 2, \ldots, n.$$

The idea of ratio criterion begins with determining the factor γ by Eqs.(6) and (7), and then does a parametric transformation $\hat{t}=t/[t+\gamma(1-t)]$ or $t=\gamma\hat{t}/[\gamma\hat{t}+(1-\hat{t})]$ so that the weights become $\hat{\omega}_i=\gamma^i\omega_i$. It requires a series of transformations step by step and finally gives rise to an LP problem, which has a direct solution. Compared with it, the idea of variance criterion is more straightforward, which adopts the concept of variance from statistics; furthermore, the procedure is very simple, only requiring finding all roots of an equation of the $(3n-2)$th degree.

Combining the idea of ratio criterion for reparameterization with Algorithm 1, we then obtain

Algorithm 3 First reparameterize the rational Bézier curve (2) by ratio criterion, and then do constrained multi-degree reduction of the curve in homogeneous coordinates. Thus, each component in homogeneous coordinates of the control points of the degree reduced curve can be given by Eq.(5), where $\hat{\omega}_i = \gamma^i \omega_i$ (i=1, 2, …, n), and γ is determined by Eqs.(6) and (7).

COMPARISONS BETWEEN FOUR DEGREE REDUCTION ALGORITHMS FOR RATIONAL BÉZIER CURVES

In this section, we will show some practical examples for Algorithms 2 and 3, and compare them with "cancelling the best linear common divisor" (Sederberg and Chang, 1993) and "shifted Chebyshev polynomial" (Chen, 1994).

Because the four degree reduction algorithms all have quite good approximating effect, to show their differences clearly, the control polygons of rational Bézier curves in the following examples are omitted; and for convenience of writing, all rational Bézier curves are presented in sequence of point array (x_i, y_i, ω_i), where (x_i, y_i) are affine coordinates of the control points and ω_i are weights of the control points.

The comparisons between the four algorithms mentioned above are divided into two parts: the first part is the comparison between the algorithm of variance criterion and ratio criterion; the second is comparing the algorithm of variance criterion with "cancelling the best linear common divisor" and "shifted Chebyshev polynomial".

Comparison between algorithms of variance criterion and ratio criterion

Example 1 Given the point array of a 4-degree rational Bézier curve: (0, 0, 1), (2, 2, 4), (3, 0, 2), (4, −2, 1), (4, 0, 1). Apply the two criterion algorithms to obtain 1-dgree reduced curves, which preserve (0, 0) degrees continuity at the end points. The result is shown in Fig.1a.

The procedures of Algorithm 2 as given below firstly achieves c=0.6604 from Eqs.(3) and (4), then we have:

$$\boldsymbol{G}_{4\times5}^{4,3,1,1} = \begin{pmatrix} 1.0000 & 0 & 0 & 0 & 0 \\ -0.2619 & 1.0476 & 0.4286 & -0.2857 & 0.0714 \\ 0.0714 & -0.2857 & 0.4286 & 1.0476 & -0.2619 \\ 0 & 0 & 0 & 0 & 1.0000 \end{pmatrix},$$

$$\hat{\boldsymbol{A}} = \begin{pmatrix} \hat{\omega}_0 x_0 & \hat{\omega}_0 y_0 & \hat{\omega}_0 \\ \hat{\omega}_1 x_1 & \hat{\omega}_1 y_1 & \hat{\omega}_1 \\ \cdots & \cdots & \cdots \\ \hat{\omega}_4 x_4 & \hat{\omega}_4 y_4 & \hat{\omega}_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0.1902 \\ 2.3042 & 2.3042 & 1.1521 \\ 2.6168 & 0 & 0.8723 \\ 2.6416 & -1.3208 & 0.6604 \\ 4.0000 & 0 & 1.0000 \end{pmatrix},$$

$$\overline{\boldsymbol{A}} = \boldsymbol{G}_{4\times5}^{4,3,1,1} \hat{\boldsymbol{A}} = \begin{pmatrix} 0 & 0 & 0.1902 \\ 3.0665 & 2.7914 & 1.4138 \\ 2.1830 & -2.0421 & 0.4882 \\ 4.0000 & 0 & 1.0000 \end{pmatrix}.$$

Finally, we get the point array of the degree reduced curve: (0, 0, 0.1902), (2.1690, 1.9744, 1.4138), (4.4715, −4.1829, 0.4882), (4.0000, 0, 1.0000).

Example 2 Given the point array of a 5-degree rational Bézier curve: (0, 0, 1), (2, 10, 2), (6, 12, 4), (10, 8, 7), (7, 1, 2), (6, 2, 3). Apply the two criterion algorithms to obtain 1-dgree reduced curves, which preserve (0, 1) degrees continuity at the end points. The result is shown in Fig.1b.

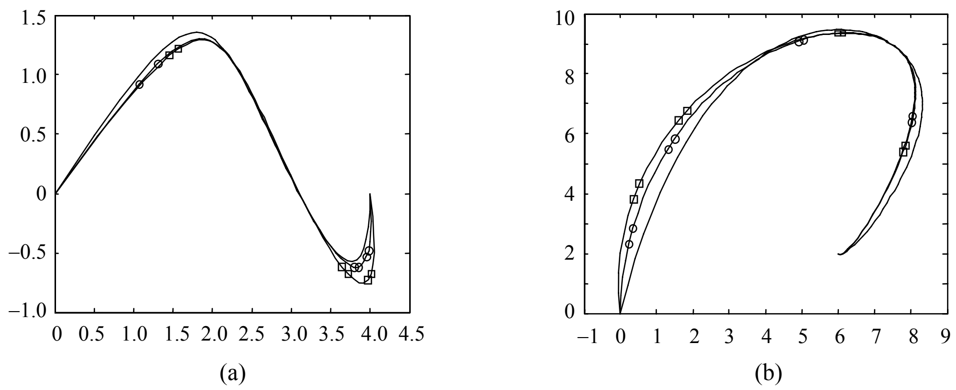

Fig.1 Comparison between two criterion algorithms. Solid line without marks: the original curve; marked by squares: ratio criterion; by circles: variance criterion. (a) Degree 4 to degree 3; (b) Degree 5 to degree 4

From these two examples, we can see that the degree reduced curves given by the two criterion algorithms have good approximating effects. And the variance criterion algorithm is the better one.

Comparison between variance criterion algorithm and two traditional degree reduction algorithms

It is obvious that the degree reduced curves given by "cancelling the best linear common divisor" cannot satisfy the end points continuity conditions; even by "shifted Chebyshev polynomial", the degree reduced curves can merely preserve the end points with (0, 0) degrees continuity. Whereas the variance criterion algorithm does not have such restriction and the degree reduced curves can preserve high degrees continuity at the end points.

Secondly, neither "cancelling the best linear common divisor" nor "shifted Chebyshev polynomial" can do multi-degree reduction at one time, which are time-consuming and usually lead to exorbitant accumulated errors; whereas that is one of the advantages of the variance criterion algorithm.

Moreover, Algorithm 2 shows that the degree reduction based on variance criterion is virtually to multiply a column vector generated by the control points of the reparameterized rational curve by a matrix. Here the matrix is independent of the original curve and hence can be calculated beforehand for invoking at any time. This guarantees the speed and simplicity of the degree reduction.

As a result, the variance criterion algorithm is superior in function to the two traditional algorithms.

In the following examples, we want to compare the degree reduction error of these three algorithms. Because of the limited function, the demand for end points continuity in the following examples is avoided for the algorithm of "cancelling the best linear common divisor". On the other hand, if multi-degree reduction is demanded for the variance criterion algorithm, for the other two, it is degree reduced stepwise. The results of numerical experiments showed that the variance criterion algorithm is superior to the two traditional algorithms in approximation precision.

Example 3 Given the point array of a 4-degree rational Bézier curve: (0, 0, 1), (2, 2, 4), (3, 0, 2), (4, −2, 1), (4, 0, 1) [the data is taken from (Sederberg and Chang, 1993)]. Apply the three algorithms to obtain 2-degree reduced curves, which preserve (0, 0) degrees continuity at the end points. The result is shown in Fig.2a.

Example 4 Given the point array of a 5-degree rational Bézier curve: (0, 0, 1), (3, 11, 3), (4, 6, 17), (4, 3, 12), (5, 7, 15), (5, 5, 20). Apply the three degree reduced algorithms to obtain 1-degree reduced curves, which preserve (0, 0) degrees continuity at the end points. The result is shown in Fig.2b.

Example 5 Given the point array of a 5-degree rational Bézier curve: (0, 0, 1), (2, 10, 3), (4, 6, 9), (6, 6, 12), (7, 10, 20), (8, 1, 30). Apply the three degree reduced algorithms to obtain 2-degree reduced curves, which preserve (0, 0) degrees continuity at the end points. The result is shown in Fig.2c.

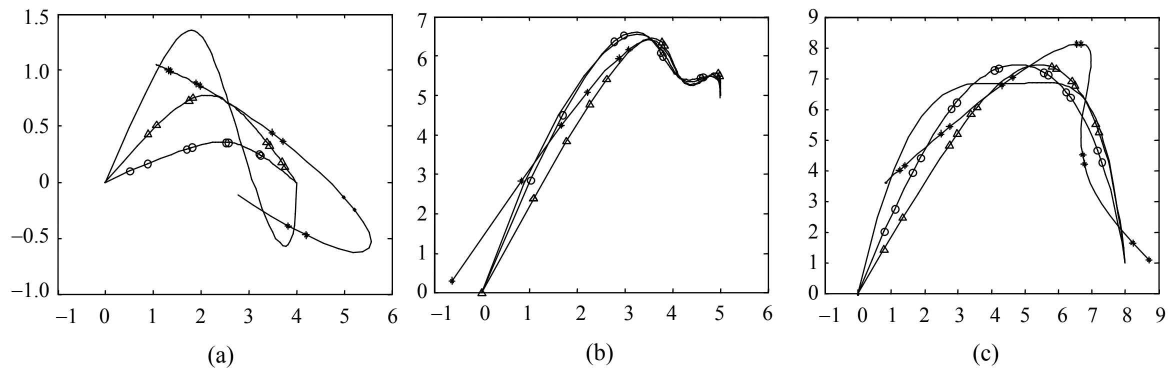

Fig.2 Comparison between three degree reduced algorithms. Solid line without marks: the original curve; marked by circles: variance criterion; by asterisks: cancelling the best linear common divisor; by triangles: shifted Chebyshev polynomial. (a) Degree 4 to degree 2; (b) Degree 5 to degree 4; (c) Degree 5 to degree 3

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