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Non-formation of vacuum states for Navier-Stokes equations with density-dependent viscosity*

ZHANG Ting, FANG Dao-yuan‡
(*Department of Mathematics, Zhejiang University, Hangzhou 310027, China*)
E-mail: zhangting79@hotmail.com; dyf@zju.edu.cn
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Abstract: We consider the Cauchy problem, free boundary problem and piston problem for one-dimensional compressible Navier-Stokes equations with density-dependent viscosity. Using the reduction to absurdity method, we prove that the weak solutions to these systems do not exhibit vacuum states, provided that no vacuum states are present initially. The essential requirements on the solutions are that the mass and energy of the fluid are locally integrable at each time, and the $L_{\mathrm{loc}}{}^1$-norm of the velocity gradient is locally integrable in time.

Key words: Compressible Navier-Stokes equations, Vacuum states, Density-dependent viscosity
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INTRODUCTION

The Navier-Stokes equations express the conservation of mass and the balance of momentum as follows:

$$\rho_t+(\rho u)_x=0, \tag{1}$$
$$(\rho u)_t+(\rho u^2+P)_x=(\mu u_x)_x+\rho f,\quad (x,t)\in\mathbb{R}\times\mathbb{R}_+, \tag{2}$$

where ρ, u, P, μ and f denote the density, velocity, pressure, viscosity coefficient and external force, respectively. We assume that $P=P(\rho,x,t)$ and $\mu=\mu(\rho,x,t)$ satisfying

$$P(0,x,t)=0,\quad x\in\mathbb{R},\ 0\le t\le T, \tag{3}$$
$$\mu(0,x,t)=\underline{\mu}>0,\quad x\in\mathbb{R},\ 0\le t\le T, \tag{4}$$

where $T>0$ is a positive time which will be fixed throughout the paper. Considering the case that $P=P(\rho)$ and $\mu=\mu(\rho)$, Zhang (2006) proved a global existence result of the weak solution to the 1D compressible isentropic Navier-Stokes equations. For the external force f in this paper, we assume only that

$$f\in L^1([0,T];L^\infty_{\mathrm{loc}}(\mathbb{R})). \tag{5}$$

In physics and mathematics, an interesting question is whether the density develops vacuum states in finite time. Hoff and Smoller (2001) considered the case that the viscosity coefficient is a positive constant, and proved that weak solutions of the Cauchy problem for the Navier-Stokes equations do not exhibit vacuum states, provided that no vacuum states are present initially. For the spherically symmetric case, see (Xin and Yuan, 2006). Indeed, the viscosity μ of the real gas varies with the density (Jiang, 1994). In this paper, we prove the similar results for weak solutions to the Cauchy problem, free boundary problem and piston problem for the Navier-Stokes equations with density-dependent viscosity.

First, we consider the Cauchy problem. The weak solution of the Cauchy problem is defined in the usual way: we say that (ρ, u) is a weak solution of Eqs.(1) and (2) on $\mathbb{R}\times[0,T]$ provided that

‡ Corresponding author
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(H1) ρ and ρu are in $C([0,T];\ H^{-1}_{\rm loc}(\mathbb{R}))$ with ρ being nonnegative; $\rho(\cdot,t)$ and $\rho u(\cdot,t)$ are in $L^1_{\rm loc}(\mathbb{R})$ for each $t\in[0,T]$; ρu^2, $P(\rho,\cdot,\cdot)$ and μu_x are in $L^1([-L,L]\times[0,T])$ for any $L>0$; (ρ, u) satisfies

$$\int \rho\varphi\Big|_{t_2}^{t_1}\mathrm{d}x=\int_{t_2}^{t_1}\int[\rho\varphi_t+\rho u\varphi_x]\mathrm{d}x\mathrm{d}t, \tag{6}$$

and

$$\int \rho u\varphi\Big|_{t_2}^{t_1}\mathrm{d}x=\int_{t_2}^{t_1}\int[\rho u(\varphi_t+u\varphi_x)+ (P-\mu u_x)\varphi_x+\rho f\varphi]\mathrm{d}x\mathrm{d}t \tag{7}$$

for all test function $\varphi\in C^1_c(\mathbb{R}^2)$, $t_1, t_2\in[0,T]$.

It follows as a consequence of (H1) that, for any $L>0$, there is a positive constant $C(L)$ such that

$$\int_{-L}^{L}\rho(x,t)\mathrm{d}x\le C(L),\ \forall t\in[0,T]. \tag{8}$$

Furthermore, we assume that

(H2) $u_x\in L^1([0,T];\ L^1_{\rm loc}(\mathbb{R}))$. There is a function $\gamma(t)\in L^1([0,T])$ such that, for all $L>0$ and almost all $t\in[0,T]$,

$$\sqrt{\int_{-L}^{L}\rho u^2\mathrm{d}x}\le\gamma(t)(1+L), \tag{9}$$

and

$$\int_{-L}^{L}|u_x|\mathrm{d}x\le\gamma(t)(1+L). \tag{10}$$

For any $L>0$, there is a positive constant $C(L)$ such that

$$\int_{-L}^{L}|\rho u|\mathrm{d}x\le C(L),\quad \forall t\in[0,T]. \tag{11}$$

(H3) There is a nonnegative continuous "potential energy density" function $G(\rho,x,t)$ on $\mathbb{R}_+\times\mathbb{R}\times[0,T]$, such that

(H3.1) there exist two positive constants C_0 and $\underline{\rho}$ such that, for all $(x,t)\in\mathbb{R}\times[0,T]$ and $\rho\in[0,\underline{\rho}]$,

$$C_0G(\rho,x,t)\ge 1. \tag{12}$$

(H3.2) there exist two constants $C_1>0$ and $\theta\in[0,1)$ such that, for all $x_0\in\mathbb{R}$, $L>0$ and $t\in[0,T]$,

$$C_0\int_{x_0}^{x_0+L}G(\rho,x,t)\mathrm{d}x\le C_0C_1+\theta L. \tag{13}$$

Remark 1 The conditions (H1)~(H3) are modified from (Hoff and Smoller, 2001). Here, we only require that the $L^1_{\rm loc}$-norm of the velocity gradient is integrable in time, which is a somewhat weaker requirement than that made in (Hoff and Smoller, 2001).

Theorem 1 Assume that P, μ and f satisfy conditions (3)~(5), let (ρ, u) be a solution of Eqs.(1) and (2) on $[0,T]$ satisfying assumptions (H1)~(H3). If $\int_E\rho(x,0)\mathrm{d}x>0$ for every open subset $E\subset\mathbb{R}$, then $\int_F\rho(x,t)\mathrm{d}x>0$ for every open subset $F\subset\mathbb{R}$ and $t\in[0,T]$.

Remark 2 In (Hoff and Smoller, 2001), the arguments need some small adjustments. In Page 273 of their paper, $\bar{u}+\varepsilon\le\|u\|_{L^\infty(Bjk)}$ does not contradict Eq.(2.36) if $\bar{u}<0$. Duan and Zhao (2005) first pointed out this one. In addition, we must point out another one: from Lemma 2.2 of (Hoff and Smoller, 2001), one can only obtain that $\rho(\cdot,s')=0$ a.e. on

$$\left(y(s)+\int_s^{s'}\|u\|_{L^\infty(y(s),z(s))}\mathrm{d}t,\ \ z(s)-\int_s^{s'}\|u\|_{L^\infty(y(s),z(s))}\mathrm{d}t\right),$$

since $(y(s), z(s))\subset(-w(t), w(t))$ does not hold in general. We include these modifications in our proof.

Using similar arguments in the proof of Theorem 1, we could obtain the following result, and omit the details of the proof.

Theorem 2 Assume that P, μ and f satisfy conditions (3), (5) and $\mu(0, x, t)=0$. Let (ρ, u) be a solution of Eqs.(1) and (2) on $[0,T]$ satisfying assumptions (H1)~(H3) and satisfying $u_x\in L^1(0,T;\ L^\infty_{\rm loc}(\mathbb{R}))$ and $u_{xx}=0$ in $D'(V)$, where $V=\{(x, t)|\rho(x, t)=0\}$. If $\int_E\rho(x,0)\mathrm{d}x>0$ for every open subset $E\subset\mathbb{R}$, then $\int_F\rho(x,t)\mathrm{d}x>0$ for every open subset $F\subset\mathbb{R}$ and $t\in[0,T]$.

Free boundary problem

Using similar arguments in the proof of Theorem 1, we could obtain the same result for the following free boundary problem:

$$\rho_t+(\rho u)_x=0, \tag{14}$$

$$(\rho u)_t+(\rho u^2+P)_x=(\mu u_x)_x+\rho f,\ \ (x,t)\in(b(t),\infty)\times\mathbb{R}_+, \tag{15}$$

where $b(t)$ is the free boundary satisfying $b(0)=b_0$, $b'(t)=u(b(t),t)$, with the boundary condition being

$$(P-\mu u_x)|_{x=b(t)}=P_\Gamma(t). \tag{16}$$

Here, b_0 is a given constant and P_Γ is the external pressure satisfying $P_\Gamma\in L^1([0,T])$. We say that (ρ, u, b) is a weak solution of Eqs.(14)~(16) provided that

(F1) $b\in C([0,T])$, $u\in L^1([0,T];C[b(t),b(t)+L])$, $\forall L>0$, the equality $b(t)=b_0+\int_0^t u(b(s),s)\mathrm{d}s$ holds $\forall t\in[0,T]$; ρ and ρu are in $C([0,T];\ H^{-1}_{\mathrm{loc}}([b(t),\infty)))$ with ρ being nonnegative; $\rho(\cdot,t)$ and $\rho u(\cdot,t)$ are in $L^1_{\mathrm{loc}}([b(t),\infty))$ for each $t\in[0,T]$; ρu^2, $P(\rho,\cdot,\cdot)$ and μu_x are in $L^1([b(t),b(t)+L]\times[0,T])$, $\forall L>0$; (ρ, u, b) satisfies

$$\int_{b(t)}^{\infty}\rho\varphi\mathrm{d}x\Big|_{t_2}^{t_1}=\int_{t_2}^{t_1}\int_{b(t)}^{\infty}[\rho\varphi_t+\rho u\varphi_x]\mathrm{d}x\mathrm{d}t,$$

and

$$\int_{b(t)}^{\infty}\rho u\varphi\mathrm{d}x\Big|_{t_2}^{t_1}=\int_{t_2}^{t_1}\int_{b(t)}^{\infty}[\rho u(\varphi_t+u\varphi_x)+(P-\mu u_x)\varphi_x+\rho f\varphi]\mathrm{d}x\mathrm{d}t+\int_{t_2}^{t_1}P_\Gamma\varphi(b(t),t)\mathrm{d}t$$

for all test function $\varphi\in C_c^1(\mathbb{R}^2)$, $t_1, t_2\in[0,T]$.

(F2) There is a function $\gamma(t)\in L^1([0,T])$ such that, for all $L>0$ and almost all $t\in[0,T]$,

$$\sqrt{\int_{b(t)}^{b(t)+L}\rho u^2\mathrm{d}x}\le\gamma(t)(1+L),$$

and

$$\int_{b(t)}^{b(t)+L}|u_x|\mathrm{d}x\le\gamma(t)(1+L).$$

For any $L>0$, there is a positive constant $C(L)$ such that

$$\int_{b(t)}^{b(t)+L}|\rho u|\mathrm{d}x\le C(L),\ \forall t\in[0,T].$$

(F3) $G(\rho, x, t)$ satisfies

(F3.1) there exist two positive constants C_0 and $\underline{\rho}$ such that $C_0G(\rho,x,t)\ge1$, for all $(x,t)\in[b(t),\infty)\times[0,T]$ and $\rho\in[0, \underline{\rho}]$;

(F3.2) there exist two constants $C_1>0$ and $\theta\in[0,1)$ such that $C_0\int_{x_0}^{x_0+L}G(\rho,x,t)\mathrm{d}x\le C_0C_1+\theta L$, for all $(x_0,t)\in[b(t),\infty)\times[0,T]$ and $L>0$.

Theorem 3 Assume that P, μ and f satisfy conditions (3)~(5), let (ρ,u) be a solution of Eqs.(14)~(16) on $[0,T]$ satisfying assumptions (F1)~(F3). If $\int_E\rho(x,0)\mathrm{d}x>0$ for every open subset $E\subset[b_0,\infty)$, then $\int_F\rho(x,t)\mathrm{d}x>0$ for every open subset $F\subset[b(t),\infty)$ and $t\in[0,T]$.

Remark 3 For the free boundary problem of 1D isentropic fluids with degenerate viscosity, see (Zhang and Fang, 2006) and the references therein.

Piston problem

Using similar arguments in the proof of Theorem 1, we also can obtain the same result for the following piston problem:

$$\rho_s+(\rho v)_y=0, \tag{17}$$
$$(\rho v)_s+(\rho v^2+P)_y=(\mu u_y)_y,\ (y,s)\in(b(s),\infty)\times\mathbb{R}_+, \tag{18}$$

where $b(s)$ is the free boundary satisfying $b(0)=b_0$, $b'(s)=v^0(s)$; $v^0\in L^1([0,T])$ is the given velocity of the piston, and the boundary condition is

$$v|_{y=b(s)}=v^0(s). \tag{19}$$

Using the transformation $x=y-\int_0^s v_0\mathrm{d}s'$, $t=s$, $u=v-v^0$, we could transform Eqs.(17)~(19) to

$$\rho_t+(\rho u)_x=0, \tag{20}$$
$$(\rho u)_t+(\rho u^2+P)_x=(\mu u_x)_x,\ (x,t)\in(0,\infty)\times\mathbb{R}_+, \tag{21}$$
$$u(0,t)=0. \tag{22}$$

We say that (ρ, u) is a weak solution of Eqs.(20)~(22) provided that

(P1) ρ and ρu are in $C([0,T];H^{-1}_{\mathrm{loc}}([0,\infty)))$ with ρ being nonnegative; $\rho(\cdot,t)$ and $\rho u(\cdot,t)$ are in $L^1_{\mathrm{loc}}([0,\infty))$ for each $t\in[0,T]$; ρu^2, $P(\rho,\cdot,\cdot)$ and μu_x are in $L^1([0, L]\times[0,T])$ for any $L>0$; (ρ, u) satisfies

$$\psi u\in L^1(0,T;W_0^{1,1}(0,\infty)),$$
$$\int_0^\infty\rho\psi\Big|_{t_2}^{t_1}\mathrm{d}x=\int_{t_2}^{t_1}\int_0^\infty[\rho\psi_t+\rho u\psi_x]\mathrm{d}x\mathrm{d}t,$$

and

$$\int_0^\infty\rho u\varphi\Big|_{t_2}^{t_1}\mathrm{d}x=\int_{t_2}^{t_1}\int_0^\infty[\rho u(\varphi_t+u\varphi_x)+(P-\mu u_x)\varphi_x]\mathrm{d}x\mathrm{d}t$$

for all test functions $\psi \in C_c^1(\mathbb{R}^2)$, $\varphi \in C_c^1(\mathbb{R}_+ \times \mathbb{R})$, $t_1, t_2 \in [0,T]$.

(P2) There is a function $\gamma(t) \in L^1([0,T])$ such that, for all $L>0$ and almost all $t \in [0,T]$,

$$\sqrt{\int_0^L \rho u^2 \mathrm{d}x} \le \gamma(t)(1+L),$$

and

$$\int_0^L |u_x| \mathrm{d}x \le \gamma(t)(1+L).$$

For any $L>0$, there is a positive constant $C(L)$ such that

$$\int_0^L |\rho u| \mathrm{d}x \le C(L), \ \forall t \in [0,T].$$

(P3) $G(\rho, x, t)$ satisfies

(P3.1) there exist two positive constants C_0 and $\underline{\rho}$ such that $C_0 G(\rho,x,t) \ge 1$, for all $(x,t) \in [0,\infty) \times [0,T]$ and $\rho \in [0, \underline{\rho}]$;

(P3.2) there exist two constants $C_1>0$ and $\theta \in [0,1)$ such that $C_0 \int_{x_0}^{x_0+L} G(\rho,x,t)\mathrm{d}x \le C_0 C_1 + \theta L$, for all $(x_0,t) \in [0,\infty) \times [0,T]$ and $L>0$.

Theorem 4 Assume that P and μ satisfy conditions (3) and (4), let (ρ, u) be a solution of Eqs.(20)~(22) on $[0,T]$ satisfying assumptions (P1)~(P3). If $\int_E \rho(x,0)\mathrm{d}x > 0$ for every open subset $E \subset [0,\infty)$, then $\int_F \rho(x,t)\mathrm{d}x > 0$ for every open subset $F \subset [0,\infty)$ and $t \in [0,T]$.

In this paper, by using some precise estimates, we obtain a result under the general case that the viscosity coefficient depends on the density. Therefore, our procedure is very similar to that of (Hoff and Smoller, 2001), and many proofs of the lemmas are just modified to our case, but for the convenience of the reader, we still give the details.

PROOF OF THEOREM 1

Throughout this section, C will denote a universal positive constant whose precise meaning will be clear from the context.

Lemma 1 Under the assumptions of Theorem 1, we have $u \in L^1([0,T]; L^\infty_{\mathrm{loc}}(\mathbb{R}))$. In fact, there is a constant $C_2>0$ such that for any $L>0$,

$$\| u(\cdot,t) \|_{L^\infty(-L,L)} \le C_2 \gamma(t)(1+L) \tag{23}$$

for almost all $t \in [0,T]$.

Proof From assumption (H2), $u_x(\cdot,t) \in L^1_{\mathrm{loc}}$, for almost all $t \in [0,T]$. Now we have to prove that Eq.(23) holds true $\forall t \in \{s \in [0,T] | u_x(\cdot,s) \in L^1_{\mathrm{loc}}\}$. Define

$$A_{l,a} = \{x \in [a,a+l] | \rho(x,t) \le \underline{\rho}\}$$

for any given $l>0$ and $a \in [-L,L]$. Since Eq.(12) implies that $C_0 G(\rho,x,t) \ge 1$ if $0 \le \rho \le \underline{\rho}$, we have $meas(A_{l,a}) \le C_0 C_1 + \theta l$ by using Eq.(13), and $meas(A_{l,a}) \le (1+\theta)l/2$ if one takes l such that $C_0 C_1 + \theta l \le (1+\theta)l/2$. Thus if $B_{l,a} = [a,a+l] - A_{l,a}$, then $meas(B_{l,a}) \ge (1-\theta)l/2$. Now if $b \in B_{l,a}$, then $\rho(b,t) \ge \underline{\rho}$ and

$$|u(a,t)| \le \underline{\rho}^{-1/2} |\sqrt{\rho} u|(b,t) + \int_a^{a+l} |u_x| \mathrm{d}x.$$

Integrating it with respect to b over $B_{l,a}$ gives

$$|u(a,t)| \le C(1+L+l)\gamma(t).$$

We shall show that, if a vacuum state was to occur, an infinite impulse would be imposed on the adjacent fluid, thus violating the assumption that the momentum is locally finite.

Remark 4 If $\rho(\cdot,t)=0$ a.e. on an open interval (a,b), then $b-a$ is bounded above by a constant depending only on the parameters C_0, C_1 and θ. Indeed, it follows from Eqs.(12) and (13) that $C_1 + \theta C_0^{-1}(b-a) \ge C_0^{-1}(b-a)$ and $b-a \le C$.

The following lemma shows that if $\rho(\cdot,t)=0$ a.e. on an interval, then, if t' is near t, $\rho(\cdot,t')=0$ a.e. on the nearby.

Lemma 2 Let $t_1 \in (0,T)$, suppose that $\rho(\cdot,t_1)=0$ a.e. on an open interval (a,b) with $a_1 \le a \le a_2 < b_2 \le b \le b_1$. Let

$$t_0 = \inf\left\{ t \in [0,t_1] \,|\, \int_t^{t_1} \| u(\cdot,s) \|_{L^\infty(a',b')} \mathrm{d}s < (b_2 - a_2)/2 \right\},$$

$$t_2 = \sup\left\{ t \in [t_1,T] \,|\, \int_{t_1}^{t} \| u(\cdot,s) \|_{L^\infty(a',b')} \mathrm{d}s < (b_2 - a_2)/2 \right\},$$

where $a'=(3a_1-b_1)/2$ and $b'=(3b_1-a_1)/2$. Under the assumptions of Theorem 1, we have for any $t\in(t_1, t_2)$, $\rho(\cdot,t)=0$ a.e. on the interval

$$\left(a+\int_{t_1}^{t}\sup_{x\in(a',b')}u(x,s)\mathrm{d}s,\ b+\int_{t_1}^{t}\inf_{x\in(a',b')}u(x,s)\mathrm{d}s\right):=I_1,$$

and for any $t\in(t_0, t_1)$, $\rho(\cdot,t)=0$ a.e. on the interval

$$\left(a-\int_{t}^{t_1}\inf_{x\in(a',b')}u(x,s)\mathrm{d}s,\ b-\int_{t}^{t_1}\sup_{x\in(a',b')}u(x,s)\mathrm{d}s\right).$$

Proof From Lemma 1, we have $t_0<t_1<t_2$. Now suppose $t>t_1$ (the proof for $t<t_1$ is similar and will be omitted herein). Fix $\delta>0$ satisfying $6\delta<b_2-a_2$, and for small $\varepsilon>0$, let u^ε denote the usual spatial regularization of u. Then for almost all $t\in[t_1,T)$,

$$\|u^\varepsilon(\cdot,t)\|_{L^\infty(a'+\delta,b'-\delta)}\le\|u(\cdot,t)\|_{L^\infty(a',b')}.$$

Letting $\inf u^\varepsilon=\inf_{(a'+\delta,\,b'-\delta)}u^\varepsilon$, $\sup u^\varepsilon=\sup_{(a'+\delta,\,b'-\delta)}u^\varepsilon$, $\|u\|_\infty=\|u(\cdot,t)\|_{L^\infty(a',b')}$, one can easily get

$$-\infty<-\|u\|_\infty\le\inf u^\varepsilon\le\sup u^\varepsilon\le\|u\|_\infty<\infty.$$

Now define a smooth function $w^{\varepsilon\delta}$ by

$$w^{\varepsilon\delta}(x,t)=\begin{cases}\sup u^\varepsilon, & x\le(a_2+b_2)/2-\delta,\\ \inf u^\varepsilon, & x\ge(a_2+b_2)/2+\delta,\end{cases}$$

$w^{\varepsilon\delta}(x,t)$ is decreasing on $((a_2+b_2)/2-\delta, (a_2+b_2)/2+\delta)$, and another smooth function $\Psi^\delta(x)$ by

$$\Psi^\delta(x)=\begin{cases}0, & x\le a+\delta,\\ 1, & a+2\delta\le x\le b-2\delta,\\ 0, & x\ge b-\delta.\end{cases}$$

$\Psi^\delta(x)$ is strictly increasing on $(a+\delta, a+2\delta)$ and strictly decreasing on $(b-2\delta, b-\delta)$.

Now let $\Phi^{\varepsilon\delta}$ be a solution of the problem

$$\begin{cases}\Phi_t+w^{\varepsilon\delta}\Phi_x=0,\ t>t_1,\\ \Phi(\cdot,t_1)=\psi^\delta.\end{cases}\tag{24}$$

By the characteristic method, it is easy to check that $\Phi^{\varepsilon\delta}(x,t)$ satisfies

$$\begin{cases}\Phi^{\varepsilon\delta}(x,t)=0, & x\le x_1(t),\\ \partial_x\Phi^{\varepsilon\delta}(x,t)>0, & x_1(t)<x<x_2(t),\\ \Phi^{\varepsilon\delta}(x,t)=1, & x_2(t)\le x\le x_3(t),\\ \partial_x\Phi^{\varepsilon\delta}(x,t)<0, & x_3(t)<x<x_4(t),\\ \Phi^{\varepsilon\delta}(x,t)=0, & x\ge x_4(t),\end{cases}\tag{25}$$

when $t\in[t_1,T^{\varepsilon\delta}]$ for some constant $T^{\varepsilon\delta}$, which is defined later. Here $x=x_i(t)$ $(i=1,2,3,4)$ are the characteristics passing through $(a+\delta, t_1)$, $(a+2\delta, t_1)$, $(b-2\delta, t_1)$, $(b-\delta, t_1)$ respectively.

Now define

$$T^{\varepsilon\delta}=\sup\{t\in[t_1,T]|(a_2+b_2)/2-x_2(s)\ge\delta,\ x_3(s)-(a_2+b_2)/2\ge\delta,\ \forall s\in[t_1,t]\},$$

and

$$T^\delta=\sup\left\{t\in[t_1,T]\,|\int_{t_1}^{t}\|u\|_\infty\,\mathrm{d}s<(b_2-a_2)/2-3\delta\right\}.\tag{26}$$

Claim 1 $T^{\varepsilon\delta}\ge T^\delta$.

If not, we have $T^{\varepsilon\delta}<T^\delta$ and

$$\int_{t_1}^{T^{\varepsilon\delta}}\|u\|_\infty\,\mathrm{d}s<(b_2-a_2)/2-3\delta.$$

Since the characteristics of Eq.(24) are given by $x'(t)=w^{\varepsilon\delta}$, $t\in[t_1,T^{\varepsilon\delta}]$, we have

$$(a_2+b_2)/2-x_2(s)>\delta,\ x_3(s)-(a_2+b_2)/2>\delta,\ s\in[t_1,T^{\varepsilon\delta}].$$

It contradicts the definition of $T^{\varepsilon\delta}$, thus Claim 1 holds.

From Eq.(25), we have that the support of $\Phi^{\varepsilon\delta}$ is the region bounded by the characteristics x_1 and x_4. As before, by the characteristic method, we obtain that $\mathrm{spt}\,\Phi^{\varepsilon\delta}(\cdot,t)$=closure of $I^{\varepsilon\delta}$, $t\in[t_1, T^\delta]$, where

$$I^{\varepsilon\delta}=\left(a+\delta+\int_{t_1}^{t}\sup u^\varepsilon\mathrm{d}s,\ b-\delta+\int_{t_1}^{t}\inf u^\varepsilon\mathrm{d}s\right).$$

Noticing that, $\forall t\in[t_1, T^\delta]$, we have

$$\mathrm{spt}\,\Phi^{\varepsilon\delta}(\cdot,t)\subset(a'+\delta, b'-\delta)\subset(a', b').$$

That is, $\Phi^{\varepsilon\delta}$ is smooth, compactly supported function, and it can serve as a test function for the weak formulation of a solution of Eqs.(1) and (2). In particular, from Eq.(6) we have

$$\int_{a'+\delta}^{b'-\delta}\rho\Phi^{\varepsilon\delta}\Big|_{t_1}^{t}\,\mathrm{d}x=\int_{t_1}^{t}\int_{a'+\delta}^{b'-\delta}\rho(u-w^{\varepsilon\delta})\partial_x\Phi^{\varepsilon\delta}\mathrm{d}x\mathrm{d}s,$$

where $t\in[t_1,T^{\delta}]$. Since $\rho(x, t_1)=0$ for a.e. $x\in[a,b]$ and $\Phi^{\varepsilon\delta}(x, t_1)=0$ for $x\in(-\infty,a+\delta]\cup[b-\delta,+\infty)$, we have

$$\begin{aligned}\int_{a'+\delta}^{b'-\delta}\rho\Phi^{\varepsilon\delta}(x,t)\mathrm{d}x=&\int_{t_1}^{t}\int_{a'+\delta}^{b'-\delta}\rho(u^{\varepsilon}-w^{\varepsilon\delta})\partial_x\Phi^{\varepsilon\delta}\mathrm{d}x\mathrm{d}s\\&+\int_{t_1}^{t}\int_{a'+\delta}^{b'-\delta}\rho(u-u^{\varepsilon})\partial_x\Phi^{\varepsilon\delta}\mathrm{d}x\mathrm{d}s,\end{aligned}\tag{27}$$

where $t\in[t_1,T^{\delta}]$. From Eq.(25), if $\partial_x\Phi^{\varepsilon\delta}(x,t)<0$, then $x\geq(a_2+b_2)/2+\delta$ and $w^{\varepsilon\delta}(x,t)=\inf u^{\varepsilon}$; if $\partial_x\Phi^{\varepsilon\delta}(x,t)>0$, then $x\leq(a_2+b_2)/2-\delta$ and $w^{\varepsilon\delta}(x,t)=\sup u^{\varepsilon}$. So we have

$$\int_{t_1}^{t}\int_{a'+\delta}^{b'-\delta}\rho(u^{\varepsilon}-w^{\varepsilon\delta})\partial_x\Phi^{\varepsilon\delta}\mathrm{d}x\mathrm{d}s\leq 0,\ t\in[t_1,T^{\delta}].\tag{28}$$

Claim 2

$$\lim_{\varepsilon\to 0+}\int_{t_1}^{t}\int_{a'+\delta}^{b'-\delta}\rho(u-u^{\varepsilon})\partial_x\Phi^{\varepsilon\delta}\mathrm{d}x\mathrm{d}s=0,\ t\in[t_1,T^{\delta}].\tag{29}$$

Admit it for the moment, from Eqs.(27)~(29), we have

$$\sup\lim_{\varepsilon\to 0+}\int_{a'+\delta}^{b'-\delta}\rho\Phi^{\varepsilon\delta}(x,t)\mathrm{d}x\leq 0,\ t\in[t_1,T^{\delta}].\tag{30}$$

Thus Eq.(30) gives that $\forall t\in[t_1,T^{\delta}]$,

$$\rho(\cdot,t)=0 \text{ a.e. on } I^{\delta},$$

where

$$\begin{aligned}I^{\delta}&=\lim_{\varepsilon\to 0}I^{\varepsilon\delta}\\&=\left(a+\delta+\int_{t_1}^{t}\sup_{x\in(a'+\delta,b'-\delta)}u\mathrm{d}s,\,b-\delta+\int_{t_1}^{t}\inf_{x\in(a'+\delta,b'-\delta)}u\mathrm{d}s\right).\end{aligned}$$

If now $t<t_2$, then

$$\int_{t_1}^{t}\|u\|_{\infty}\,\mathrm{d}s<(b_2-a_2)/2,$$

and thus there is a δ_0 such that $6\delta_0<b_2-a_2$ and

$$\int_{t_1}^{t}\|u\|_{\infty}\,\mathrm{d}s<(b_2-a_2)/2-4\delta,\ \forall 0<\delta<\delta_0.$$

For such δ, Eq.(26) implies that $t\leq T^{\delta}$. Thus for such t and δ, $\rho(\cdot,t)=0$ a.e. on I^{δ}. Taking a sequence $\delta_i\to 0+$, we get that $\rho(\cdot,t)=0$ a.e. on the interval I_1, $\forall t\in[t_1, t_2)$, and this completes the proof of Lemma 2 when $t>t_1$.

It remains to prove Claim 2. To this end, we first differentiate Eq.(24) with respect to x to obtain

$$\partial_{tx}\Phi^{\varepsilon\delta}+w^{\varepsilon\delta}\partial_{xx}\Phi^{\varepsilon\delta}=-\partial_x w^{\varepsilon\delta}\partial_x\Phi^{\varepsilon\delta},$$

so that along the characteristics $x(t)=x(t; x(t_1), t_1)$,

$$\partial_x\Phi^{\varepsilon\delta}(x(t),t)=\partial_x\Psi^{\delta}(x(t_1))\exp\left(-\int_{t_1}^{t}\partial_x w^{\varepsilon\delta}(x(s),s)\mathrm{d}s\right).\tag{31}$$

Since $|\partial_x w^{\varepsilon\delta}(\cdot,s)|\leq C(\delta)\|u\|_{\infty}$, from Eq.(31), we have there is a constant $C'(\delta)$ which only depends on δ and is independent of ε, such that $\|\partial_x\Phi^{\varepsilon\delta}\|_{\infty}\leq C'(\delta)$. Hence

$$\begin{aligned}&\left|\int_{t_1}^{t}\int_{a'+\delta}^{b'-\delta}\rho(u-u^{\varepsilon})\partial_x\Phi^{\varepsilon\delta}\mathrm{d}x\mathrm{d}s\right|\\&\leq C'(\delta)\int_{t_1}^{T}\left\|(u-u^{\varepsilon})(\cdot,s)\right\|_{L^{\infty}(a',b')}\left\|\rho(\cdot,s)\right\|_{L^{1}(a',b')}\mathrm{d}s.\end{aligned}\tag{32}$$

From assumption (H2), Eqs.(8) and (23), we have $u(\cdot,t)\in W^{1,1}_{\mathrm{loc}}$ for almost all $t\in[t_1,T]$, and $\|\rho(\cdot,s)\|_{L^1(a',b')}$ is bounded $\forall s\in[t_1, T]$; thus for almost all $t\in[t_1, T]$, the integrand on the right-hand side of Eq.(32) tends to zero as $\varepsilon\to 0+$. Since $\|u(\cdot,s)\|_{L^{\infty}(a',b')}$ is integrable, applying the Lebesgue dominated convergence theorem, we show that Claim 2 holds.

Now suppose that $\rho(\cdot, t_1)=0$ a.e. on (a, b). The interval (a, b) and the time $t_1\in(0, T)$ will be fixed for the remainder of the argument. Let t_0 be as in the statement of Lemma 2 with $a_1=a=a_2$ and $b_2=b=b_1$. For $t\in(t_0, t_1)$, define

$$y(t)=\inf\{x|\rho(\cdot,t)=0 \text{ a.e. on } (x, (a+b)/2)\},$$
$$z(t)=\sup\{x|\rho(\cdot,t)=0 \text{ a.e. on } ((a+b)/2, x)\}.$$

Clearly, $y(t_1)=a$ and $z(t_1)=b$.

In the following lemma, we prove an important regularity property for the curves y and z.

Lemma 3 Under the assumptions of Theorem 1, there exists a constant $h=h(a,b)>0$ such that y and z are absolutely continuous functions on $[t_1-h, t_1]$.

Proof First, it follows from Remark 4 that there exists a constant $L_0>0$ such that

$$-L_0\le y(t)<z(t)\le L_0,\ \ \forall t\in(t_0, t_2). \tag{33}$$

Next, choose $h>0$ such that

$$4\int_{t_1-h}^{t_1}\|u\|_{L^\infty(-2L_0,2L_0)}\,\mathrm{d}t<b-a.$$

In order to prove z is absolutely continuous on $[t_1-h, t_1]$, let s and t be such that $t_1-h\le s<t\le t_1$, we will compare $z(s)$ with $z(t)$. Using Lemma 2, we have

$$-L_0\le y(t)\le(3a+b)/4<(a+3b)/4\le z(t)\le L_0,$$

and

$$-L_0\le y(s)\le(3a+b)/4<(a+3b)/4\le z(s)\le L_0.$$

Applying Lemma 2, we obtain that if $\rho(\cdot,t)=0$ on $(y(t), z(t))$ with $-L_0\le y(t)\le(3a+b)/4<(a+3b)/4\le z(t)\le L_0$, then $\rho(\cdot,s)=0$ a.e. on

$$\left(y(t)-\int_s^t\inf_{x\in(-2L_0,2L_0)}u\,\mathrm{d}\tau,\ z(t)-\int_s^t\sup_{x\in(-2L_0,2L_0)}u\,\mathrm{d}\tau\right),$$

so that

$$z(s)\ge z(t)-\int_s^t\sup_{x\in(-2L_0,2L_0)}u\,\mathrm{d}\tau. \tag{34}$$

Similarly, if $\rho(\cdot,s)=0$ on $(y(s),z(s))$, then

$$z(t)\ge z(s)+\int_s^t\inf_{x\in(-2L_0,2L_0)}u\,\mathrm{d}\tau. \tag{35}$$

Hence, we have

$$\left|z(t)-z(s)\right|\le\int_s^t\|u\|_{L^\infty(-2L_0,2L_0)}\,\mathrm{d}\tau,\ t_1-h\le s<t\le t_1. \tag{36}$$

For given $\varepsilon>0$, by Lemma 1, there is a $\delta>0$ such that if $meas(E)<\delta$, then

$$\int_E\|u\|_{L^\infty(-2L_0,2L_0)}\,\mathrm{d}\tau\le\varepsilon. \tag{37}$$

Thus, for given points $\{s_j\}$ and $\{\tau_j\}$ satisfying $t_1-h\le s_1<\tau_1<\ldots<s_k<\tau_k<t_1$ and $\sum(\tau_j-s_j)\le\delta$, Eqs.(36) and (37) give $\sum|z(\tau_j)-z(s_j)|\le\varepsilon$, and it shows that z is absolutely continuous on $[t_1-h, t_1]$. Similarly, one can show that y is absolutely continuous on the same interval.

In the next lemma, we will obtain the further results concerning the functions $y(t)$ and $z(t)$. To this end, let S be defined as the set of all $t\ge0$ such that there are extensions of y and z to $[t, t_1]$ such that the following three properties hold:

(1) y and z are absolutely continuous on $[t, t_1]$;
(2) $y<z$ on $[t, t_1]$;
(3) $\forall s\in[t, t_1]$, $\int_{y(s)-\varepsilon}^{z(s)}\rho(x,s)\mathrm{d}s$ and $\int_{y(s)}^{z(s)+\varepsilon}\rho(x,s)\mathrm{d}s$

are both positive $\forall\varepsilon>0$, and $\int_{y(s)}^{z(s)}\rho(x,s)\mathrm{d}s=0.$

From Lemma 3, we know that S is nonempty. Let $\tau=\inf S$.

Lemma 4 Under the assumptions of Theorem 1, we have that y and z have absolutely continuous extensions to time τ, $y(\tau)=z(\tau)$, and there is an $L_1>0$ such that $\forall t\in[\tau, t_1]$, $-L_1\le y(t)\le z(t)\le L_1$.

Proof We prove the last assertion first. Let $\tau<c<d<f<g<t_1$, and $w(t)=\max\{z(t),-y(t)\}$, $t\in(\tau, t_1)$. Let $t\in[c,g]$, then by the definition, we have $\rho(\cdot,t)=0$ a.e. on $(y(t), z(t))$, and $y(t)<z(t)$. Lemma 2 shows that there is $h=h(t)>0$ such that $\rho(\cdot,s)=0$ a.e. on the interval

$$\left(y(t)+\left|\int_s^t\|u\|_{L^\infty(-2w(t),2w(t))}\,\mathrm{d}\sigma\right|,\ z(t)-\left|\int_s^t\|u\|_{L^\infty(-2w(t),2w(t))}\,\mathrm{d}\sigma\right|\right),$$

$\forall|t-s|\le h$, and

$$6C_2\int_{t-h}^{t+h}\gamma(s)\mathrm{d}s\le1. \tag{38}$$

Thus we have

$$z(s)\ge z(t)-\left|\int_s^t\|u\|_{L^\infty(-2w(t),2w(t))}\,\mathrm{d}\sigma\right|,$$

and

$$y(s)\le y(t)+\left|\int_s^t\|u\|_{L^\infty(-2w(t),2w(t))}\,\mathrm{d}\sigma\right|.$$

Using Lemma 1, we get

$$w(s)\ge\left(1-2C_2\left|\int_s^t\gamma(\sigma)\mathrm{d}\sigma\right|\right)w(t)-C_2\left|\int_s^t\gamma(\sigma)\mathrm{d}\sigma\right|.$$

Thus for $|t-s|\le h$, Eq.(38) gives

$$w(t)\le\left(1+C\left|\int_s^t\gamma(\sigma)\mathrm{d}\sigma\right|\right)\left[w(s)+C\left|\int_s^t\gamma(\sigma)\mathrm{d}\sigma\right|\right]. \tag{39}$$

Now choose constants $A<B$ (depending on t, which is fixed), such that $y(t)<A<B<z(t)$. Lemma 3 implies there is $h(t)>0$ such that $y(t)-1\le y(s)<A<B<z(s)\le z(t)+1$ for all $|t-s|\le h$. For such s, using Lemma 2, we find that there is a σ (depending on t), such that if $s\le s_2\le s+\sigma$, $w(s_2)\ge w(s)-\int_s^t \|u\|_{L^\infty(-2-2w(t),2+2w(t))}\,\mathrm{d}\sigma$.

We can further reduce $h(t)$ so that $h(t)\le\sigma(t)$. Thus if $t-h(t)\le s\le t$, then $s\le t\le s+\sigma(t)$, and we may take $s_2=t$ to obtain $w(t)\ge w(s)-C_2(3+2w(t))\int_s^t \gamma(\sigma)\mathrm{d}\sigma$, where we have used Lemma 1. Thus if $t-h(t)\le s\le t$, then

$$w(s)\le\left(1+C\int_s^t \gamma(\sigma)\mathrm{d}\sigma\right)\left[w(t)+C\int_s^t \gamma(\sigma)\mathrm{d}\sigma\right]. \quad (40)$$

We now cover the interval $[d, f]$ by a finite number of intervals $B_{h_j}(s_j)$, where $s_1>s_2>\ldots>s_p$ and $h_j=h(s_j)$. From Eqs.(39) and (40), we obtain

$$w_p\le\prod\left\{\left[1+C\int_{s_{j+1}}^{\tau_j}\gamma(\sigma)\mathrm{d}\sigma\right]\left[1+C\int_{\tau_j}^{s_j}\gamma(\sigma)\mathrm{d}\sigma\right]\cdot\left[w_1+C\int_{s_p}^{s_1}\gamma(\sigma)\mathrm{d}\sigma\right]\right\},$$

where $w_p=w(s_p)$ and $w_1=w(s_1)$. Applying the inequality $\prod(1+\varepsilon_j)\le(1+\varepsilon/q)^q\le \mathrm{e}^\varepsilon$, where $\varepsilon_1+\varepsilon_2+\ldots+\varepsilon_q=\varepsilon$ and each $\varepsilon_j>0$, we have

$$w_p\le C(1+w_1). \quad (41)$$

From Eq.(33), choosing s_1 near t_1, we can bound w_1 independent of s_1. Then from Eqs.(39) and (41), we can bound $w(t)$ on $[d, t_1]$, for all $d>\tau$, independent of d. Thus we have proved that there is a constant $L_1>0$ such that $-L_1\le y(t)\le z(t)\le L_1$, $\tau<t\le t_1$.

We now show that z and y are uniformly continuous on the interval $(\tau, t_1]$. Once it is shown, the first and third assertions of Lemma 4 will be proved. To prove the uniform continuity of z on $(\tau, t_1]$, we only need to show that, for all $\varepsilon>0$, one can choose a $\delta>0$ such that if $0\le s<t\le T$ and $|s-t|\le\delta$, then

$$\int_s^t \|u\|_{L^\infty(-2L_1,2L_1)}\,\mathrm{d}\sigma\le\varepsilon.$$

To this end, we just follow the idea as that earlier in this proof, if $c\in(\tau, t_1]$, we can find $h(c)>0$ such that if $|c-d|\le h(c)$, then

$$|z(c)-z(d)|\le\left|\int_c^d \|u\|_{L^\infty(-2L_1,2L_1)}\,\mathrm{d}\sigma\right|. \quad (42)$$

For the fixed $s<t$ with $|s-t|\le\delta$ and s, $t\in(\tau, t_1]$, the interval $[s,t]$ can be covered by $\bigcup B_{h_k}(s_k)$, where $s_1<s_2<\ldots<s_p$, $s_j+h_j>s_{j+1}-h_{j+1}$ and $2h_j=h(s_j)<\delta$ for each j. Then $|s_{j+1}-s_j|<\delta$. Now for some j and k, $s\in B_{h_k}(s_k)$ and $t\in B_{h_j}(s_j)$, by Eq.(42), we have

$$|z(t)-z(s)|\le|z(s)-z(s_j)|+\ldots+|z(s_k)-z(t)|\le\varepsilon.$$

To complete the proof, we have to show that $y(\tau)=z(\tau)$. But it is clear, since otherwise $y(\tau)<z(\tau)$, if $\tau>0$, then τ would not be minimal, whereas if $\tau=0$, then the assumption $\int_{y(\tau)}^{z(\tau)}\rho(x,0)\mathrm{d}x>0$ would be violated.

We next study the function u in the vacuum region $V=\{(x,t)|y(t)<x<z(t), \tau<t\le t_1\}$.

Lemma 5 Under the assumptions of Theorem 1, there exist functions $\alpha,\beta\in L^1_{\mathrm{loc}}((\tau,t_1])$ such that $u(x,t)=\alpha(t)x+\beta(t)$, for almost all $t\in(\tau, t_1]$ and all $x\in[y(t), z(t)]$.

Proof From Eqs.(3), (4) and (7), we see that

$$\iint_V u_x\varphi_x \mathrm{d}x\mathrm{d}s=0$$

for all C^1 test function φ supported in V. Using similar arguments in (Hoff and Smoller, 2001), we can obtain this lemma easily, so details are omitted here.

Lemma 6 Fix $w_1\in(a,b)$ and define $w(t)$ by

$$w(t)=w_1\exp\left(-\int_t^{t_1}\alpha(s)\mathrm{d}s\right)-\int_t^{t_1}\exp\left(\int_s^t\alpha\right)\beta(s)\mathrm{d}s,$$

where $\tau<t\le t_1$. Under the assumptions of Theorem 1, we have $y(t)<w(t)<z(t)$ for all $\tau<t\le t_1$.

Proof We claim that

$$\frac{\mathrm{d}z}{\mathrm{d}t}\le\alpha z+\beta \quad (43)$$

for almost all $\tau<t\le t_1$. If it holds true, then since

$$\frac{\mathrm{d}w}{\mathrm{d}t}=\alpha w+\beta,\quad w(t_1)=w_1<b=z(t_1),$$

we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\exp\left(-\int_{t_1}^{t}\alpha(s)\mathrm{d}s\right)(z-w)\right]\le 0,\ \text{a.e.}$$

Integrating it from t to t_1, we have

$$\exp\left(-\int_{t_1}^{t}\alpha(s)\mathrm{d}s\right)(z-w)(t)\ge z(t_1)-w(t_1)>0.$$

Therefore $w(t)<z(t)$. Similarly, one can get $y(t)<w(t)$.

We now prove Eq.(43). Define the following sets of zero measure:

$A=\{t\in(\tau, t_1]|u_x(\cdot,t)\notin L^1(y(t),z(t))\}$,
$D=\{t\in(\tau, t_1]|u(x,t)\ne\alpha(t)x+\beta(t), x\in[y(t),z(t)]\}$,
$E=\{t\in(\tau, t_1]|z$ is not differentiable at $t\}$.

Let $B_{jk}=\{x|\, j|x-r_k|<1\}$, $j, k=1, 2, \ldots$, where $\{r_k\}$ is the set of rational numbers. From Lemma 1, we have $\sup_{x\in B_{jk}} u\in L^1([0,T])$. Let $F_{jk}=\{t\in(\tau,\ t_1)\,|\,t$ is not a Lebesgue point of $\sup_{x\in B_{jk}} u\}$, and $F=\cup F_{jk}$. Then $meas(F)=0$ and if $s\in(\tau,t_1]\setminus F$,

$$\lim_{t\to s}(t-s)^{-1}\int_s^t \sup_{x\in B_{jk}} u(x,s')\mathrm{d}s'=\sup_{x\in B_{jk}} u(x,s)$$

for every j and k.

Let $t_3\notin A\cup D\cup E\cup F$, we will prove that Eq.(43) holds at t_3. Suppose not, then there is $\varepsilon>0$ such that for t near t_3 and $t>t_3$,

$$z(t)\ge z_3+(t-t_3)(u_3+\varepsilon), \tag{44}$$

where $z_3=z(t_3)$ and $u_3=\alpha(t_3)x+\beta(t_3)$. Because $u(\cdot,t_3)\in W^{1,1}_{\text{loc}}$, we can find $h>0$ such that

$$2|u(x, t_3)-u_3|\le \varepsilon,\ \text{for all } |x-z_3|\le h, \tag{45}$$

and $y(t_3)<z_3-h$. Then choose $B_{jk}:=(c,d)$ such that

$$z_3-h<(3c-d)/2<c<z_3<d<(3d-c)/2<z_3+h,$$

and choose e such that $c<e<z_3<d$. Thus we can find $\Delta t>0$ such that

$$2\int_{t_3-\Delta t}^{t_3+\Delta t}\|u(\cdot,t)\|_{L^\infty(c',d')}\,\mathrm{d}t<e-c,$$

where $2c'=3c-d$, $2d'=3d-c$, $y(t)<c$ and $e\le z(t)\le d$ for all $|t-t_3|<\Delta t$ (this can be done since y and z are continuous functions). Then if $t\in(t_3, t_3+\Delta t)$,

$$\rho(\cdot,t)=0 \text{ a.e. on } (y(t), z(t))\supset(c, z(t)),$$

so by Lemma 2, for all $s\in(t_3-\Delta t, t)$, $\rho(\cdot,s)=0$ a.e. on

$$\left(c-\int_s^t\inf_{x\in(c',d')}u(x,s')\mathrm{d}s',\ z(t)-\int_s^t\sup_{x\in(c',d')}u(x,s')\mathrm{d}s'\right).$$

Thus, let $s=t_3$, we have

$$z(t_3)\ge z(t)-\int_s^t\sup_{x\in(c',d')}u(x,s')\mathrm{d}s'.$$

Thus using Eq.(44), we have

$$z_3+(t-t_3)(u_3+\varepsilon)\le z(t)\le z_3+\int_{t_3}^t\sup_{x\in(c',d')}u(x,s)\mathrm{d}s$$

and

$$u_3+\varepsilon\le(t-t_3)^{-1}\int_{t_3}^t\sup_{x\in(c',d')}u(x,s)\mathrm{d}s.$$

Letting $t\to t_3$ in the last inequality, we get

$$u_3+\varepsilon\le\sup_{x\in(c',d')}u(x,t_3).$$

Since $(c',d')\subset[z_3-h,\ z_3+h]$, this contradicts Eq.(45). This proves Eq.(43) and completes the proof of Lemma 6.

Corollary 1 Under the assumptions of Theorem 1, we have $\lim_{t\downarrow\tau}\int_t^{t_1}\alpha(s)\mathrm{d}s=+\infty$.

Proof With $w_1<w_2$, $w_i\in(a,b)$, $i=1,2$, from the definition of w in Lemma 6, we have

$$w_1(t)-w_2(t)=(w_1-w_2)\exp\left(-\int_t^{t_1}\alpha(s)\mathrm{d}s\right),\ \tau<t\le t_1.$$

From Lemmas 4 and 6, we have $\lim_{t\downarrow\tau}(w_1(t)-w_2(t))=0$, and the last equation gives the result.

Proof (of Theorem 1) Let $c(t)=w_1(t)<w_2(t)=d(t)$ be two curves as in Lemma 6, corresponding to points w_1 and w_2 (satisfying $a<w_1<w_2<b$), respectively. Then from Lemmas 4 and 6, we have $0\le d(t)-c(t)\to0$ as $t\downarrow\tau$. Choose $e(t_1)$ and $f(t_1)$ satisfying $d(t_1)<z(t_1)<3L_1<e(t_1)<f(t_1)$, define two smooth functions $\psi(x)$ and $\chi(x)$ by

$$\psi(x)=\begin{cases}0, & x\le c(t_1),\\ 1, & d(t_1)\le x\le e(t_1),\\ 0, & x\ge f(t_1),\end{cases}$$

and $\psi(x)$ is increasing on $(c(t_1),d(t_1))$, decreasing on $(e(t_1),f(t_1))$, $\mathrm{spt}\,\psi_x\subset(c(t_1),d(t_1))\cup(e(t_1),f(t_1))$;

$$\chi(x)=\begin{cases}0, & x\le -2L_1,\\ 1, & -L_1\le x\le L_1,\\ 0, & x\ge 2L_1,\end{cases}$$

and $\chi(x)$ is increasing on $(-2L_1, -L_1)$, decreasing on $(L_1, 2L_1)$. And define

$$w^\varepsilon(x,t)=\chi(x)(\alpha^\varepsilon(t)x+\beta^\varepsilon(t)),\quad \tau<t\le t_1,$$

where α^ε and β^ε are regularizations of α and β, respectively. Consider the initial-value problem

$$\partial_t\varphi^\varepsilon+w^\varepsilon\partial_x\varphi^\varepsilon=0,\ s\in[t, t_1), \tag{46}$$
$$\varphi^\varepsilon(x, t_1)=\psi(x), \tag{47}$$

where $\tau<t\le t_1$. By the characteristic method, it is easy to check that φ^ε is a smooth compactly supported function. The characteristics of Eqs.(46) and (47) which start on $\mathrm{spt}\,\psi_x\cap(c(t_1),d(t_1))$ are given by $x'=\alpha^\varepsilon(t)x+\beta^\varepsilon(t)$, so for small ε (depending on t) they stay between the curves $c(t)$ and $d(t)$. Similarly, the characteristics of Eqs.(46) and (47) outside of the vacuum, which issue from $[e(t_1), f(t_1)]$, are given by $x'=0$. Thus, we have, for $\tau<t\le s<t_1$,

$$\begin{cases}\partial_x\varphi^\varepsilon(x,s)\ge 0, & \text{if } (x,s)\in(c(s),d(s))\times[t,t_1]:=\mathrm{I},\\ \partial_x\varphi^\varepsilon(x,s)=\psi_x(x)\le 0, & \text{if } (x,s)\in[e(t_1),f(t_1)]\times[t,t_1]:=\mathrm{II},\\ \partial_x\varphi^\varepsilon(x,s)=0, & \text{else.}\end{cases}$$

Thus from Eq.(7), we have, for $\tau<t<t_1$,

$$\int\rho u\varphi^\varepsilon\Big|_t^{t_1}\mathrm{d}x=\iint[\rho u(u-w^\varepsilon)\partial_x\varphi^\varepsilon+(P-\mu u_x)\partial_x\varphi^\varepsilon+\rho f\varphi^\varepsilon]\mathrm{d}x\mathrm{d}s. \tag{48}$$

First, the left hand side of Eq.(48) is bounded, independent of t, for $\tau<t<t_1$, by the virtue of Eq.(11). Similarly, the term $\iint\rho f\varphi^\varepsilon\mathrm{d}x\mathrm{d}s$ is bounded, because of Eqs.(5) and (8). Also $\iint_{\mathrm{I}}\rho u(u-w^\varepsilon)\partial_x\varphi^\varepsilon\mathrm{d}x\mathrm{d}s=0$, since $\rho=0$ in I. In II, $w^\varepsilon=0$ and $\partial_x\varphi^\varepsilon=\psi_x$, so (H1) implies

$$\left|\iint_{\mathrm{II}}\rho u(u-w^\varepsilon)\partial_x\varphi^\varepsilon\mathrm{d}x\mathrm{d}s\right|=\left|\iint_{\mathrm{II}}\rho u^2\psi_x\mathrm{d}x\mathrm{d}s\right|\le C,$$

and

$$\left|\iint_{\mathrm{II}}(P-\mu u_x)\partial_x\varphi^\varepsilon\mathrm{d}x\mathrm{d}s\right|=\left|\iint_{\mathrm{II}}(P+\mu|u_x|)|\psi_x|\mathrm{d}x\mathrm{d}s\right|\le C.$$

From Eqs.(3) and (4), we have

$$\begin{aligned}\iint_{\mathrm{I}}(P-\mu u_x)\partial_x\varphi^\varepsilon\mathrm{d}x\mathrm{d}s&=-\int_t^{t_1}\int_{c(s)}^{d(s)}\underline{\mu}u_x\partial_x\varphi^\varepsilon\mathrm{d}x\mathrm{d}s\\ &=-\int_t^{t_1}\underline{\mu}\alpha(s)[\varphi^\varepsilon(d(s),s)-\varphi^\varepsilon(c(s),s)]\mathrm{d}s\\ &=-\int_t^{t_1}\underline{\mu}\alpha(s)\mathrm{d}s,\end{aligned}$$

since $\varphi^\varepsilon(d(s),s)=1$ and $\varphi^\varepsilon(c(s),s)=0$. Thus, from the arguments above, we obtain that $\left|\int_t^{t_1}\alpha(s)\mathrm{d}s\right|$ is bounded, independent of t. Letting $t\to\tau$, it contradicts Corollary 1. This completes the proof of Theorem 1.

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