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Input-output approach to robust stability and stabilization for uncertain singular systems with time-varying discrete and distributed delays*

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Abstract: Based on input-output approach, the robust stability and stabilization problems for uncertain singular systems with time-varying delays are investigated. The parameter uncertainties are assumed to be norm-bounded and the time-varying delays include both discrete delay and distributed delay. By introducing a new input-output model, the time-delay system is embedded in a family of systems with a forward system without time delay and a dynamical feedback uncertainty. A sufficient and necessary condition, which guarantees the system regular, impulse-free and stable for all admissible uncertainties, is obtained. Based on the strict linear matrix inequality, the desired robust state feedback controller is also obtained. Finally, a numerical example is provided to demonstrate the application of the proposed method.

Key words: Robust stability, Input-output stability, Singular system, Time-varying delay, Linear matrix inequality (LMI)
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INTRODUCTION

The input-output approach to robust stability, which reduces the stability analysis of the uncertain system to the analysis of a class of systems with the same nominal part but with additional inputs and outputs, has been studied in (Desoer and Vidyasagar, 1975; Vidyasagar, 1993).

On the other hand, time delays, in many real engineering systems, are often encountered and maybe a source of instability of the systems. Therefore, the analysis of time-delay systems has attracted much interest in the literature over the past several decades. Two types of stability conditions, namely delay-independent (Mahmoud, 2000; Niculoscu, 2001) and delay-dependent (Cao *et al*., 1998; Gu *et al*., 2003), have been studied. As the name implies, delay-independent results guarantee stability for arbitrarily large delays. Delay-dependent results take into account the maximum delay that can be tolerated by the system and, thus, are more useful in applications and less conservative, especially when time delays are small. However, all the aforementioned results for time-delay systems are “sufficient”, not “sufficient and necessary”. The input-output approach was introduced for constant delays in (Bliman, 2000; Huang and Zhou, 2000).

Furthermore, singular systems, which are known as descriptor systems, implicit systems, generalized state-space systems or semi-state systems, have received much attention since singular model can preserve the structure of practical systems and can better describe a large class of physical systems than state-space ones (Lewis, 1986; Dai, 1989). During recent years, much attention has been paid to the control problem of singular systems, such as stability and

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stabilization control (Xu *et al*., 2002; Zhou and Lam, 2003), H_∞ control (Masubuchi *et al*., 1997; Xu *et al*., 2003b; Yue and Han, 2005; Wang *et al*., 2007), and guaranteed cost control (Ma *et al*., 2005), etc.

However, time delays lead to additional inputs and outputs of the system. How these additional inputs and outputs influence the performance of the system is a subject of recurring interest. Therefore, the study of robust stability problem for uncertain singular systems with time delays is of theoretical and practical importance. For nonsingular systems with time delays, the input-output approach to robust stability has been investigated in (Fridman and Shaked, 2006). Compared with nonsingular systems, the study of the dynamic performance of singular systems is much more difficult than that for nonsingular systems since singular systems usually have three types of modes, namely, finite dynamics modes, impulsive modes and nondynamics modes, while the latter two do not appear in the nonsingular systems. To the authors' best knowledge, the input-output stability problem for singular systems is still open, which motivates this paper.

This paper discusses the robust stability and stabilization problems for uncertain singular systems with discrete and distributed delays. By introducing a new input-output model, a sufficient and necessary condition, which guarantees the system regular, impulse-free and stable, is obtained. The explicit expression of robust state feedback controller is also given by the form of LMI, which can be solved easily. Finally, a numerical example is given to show the effectiveness of the proposed method.

PROBLEM FORMULATION

Consider the following uncertain singular system with time-varying discrete and distributed delays:

$$\begin{aligned}\boldsymbol{E}\dot{\boldsymbol{x}}(t)&=(\boldsymbol{A}+\Delta\boldsymbol{A})\boldsymbol{x}(t)+(\boldsymbol{A}_\tau+\Delta\boldsymbol{A}_\tau)\boldsymbol{x}(t-\tau(t))+\\&\int_{-h(t)}^{0}(\boldsymbol{A}_h+\Delta\boldsymbol{A}_h)\boldsymbol{x}(t+\theta)\mathrm{d}\theta+(\boldsymbol{B}+\Delta\boldsymbol{B})\boldsymbol{u}(t),\\&\boldsymbol{x}(0)=\phi(t),\ \ t\in[-\tau',0],\ \ \tau'=\max\{\tau,h\},\end{aligned}\tag{1}$$

where $\boldsymbol{x}(t)\in\mathbb{R}^n$ and $\boldsymbol{u}(t)\in\mathbb{R}^m$ are the system state and control input, respectively; $\tau(t)$ and $h(t)$ are time-varying delays satisfying $0\le\tau(t)\le\tau<\infty$, $0\le h(t)\le h<\infty$; $\phi(t)\in\mathbb{C}_0$ represents the initial function, and $\mathbb{C}_0$ denotes the set of all continuous functions from $[-\tau', 0]$ to $\mathbb{R}^n$; $\boldsymbol{E}$, $\boldsymbol{A}$, $\boldsymbol{A}_\tau$, $\boldsymbol{A}_h$ and $\boldsymbol{B}$ are known matrices of appropriate dimensions, where $\boldsymbol{E}$ may be singular and we assume that rank($\boldsymbol{E}$)=$r\le n$. $\Delta\boldsymbol{A}$, $\Delta\boldsymbol{A}_\tau$, $\Delta\boldsymbol{A}_h$ and $\Delta\boldsymbol{B}$ denote the unknown maybe time-varying parameter uncertainties which satisfy

$$[\Delta\boldsymbol{A}\ \ \Delta\boldsymbol{A}_\tau\ \ \Delta\boldsymbol{A}_h\ \ \Delta\boldsymbol{B}]=\boldsymbol{M}\boldsymbol{F}(t)[\boldsymbol{N}_a\ \ \boldsymbol{N}_\tau\ \ \boldsymbol{N}_h\ \ \boldsymbol{N}_b],\tag{2}$$

where $\boldsymbol{M}$, $\boldsymbol{N}_a$, $\boldsymbol{N}_\tau$, $\boldsymbol{N}_h$ and $\boldsymbol{N}_b$ are known matrices of appropriate dimensions, $\boldsymbol{F}(t)$ is an unknown matrix function bounded by $\boldsymbol{F}^{\mathrm{T}}(t)\boldsymbol{F}(t)\le\boldsymbol{I}$.

Definition 1 (Xu *et al*., 2002) The uncertain singular system (1) is said to be robustly stable if the system (1) with $\boldsymbol{u}(t)=\boldsymbol{0}$ is regular, impulse-free and stable for all admissible uncertainties.

Definition 2 (Xu *et al*., 2002) The uncertain singular system (1) is said to be robustly stabilizable if there exists a linear state feedback control law $\boldsymbol{u}(t)=\boldsymbol{K}\boldsymbol{x}(t)$, $\boldsymbol{K}\in\mathbb{R}^{m\times n}$ such that the resultant closed-loop system is robustly stable in the sense of Definition 1. In this case, $\boldsymbol{u}(t)=\boldsymbol{K}\boldsymbol{x}(t)$ is said to be a robust state feedback control law for system (1).

The problem to be addressed in this paper is the development of conditions for robust stability and stabilization for the uncertain singular system (1).

Motivated by the definition of input-output stability for linear system in (Gu *et al*., 2003), we give a definition of input-output stability for the singular system.

Definition 3 (Input-Output Stability) For the following singular system:

$$\begin{cases}\boldsymbol{E}\dot{\boldsymbol{x}}(t)=\boldsymbol{f}[t,\boldsymbol{x}(t),\boldsymbol{u}(t)],\\ \boldsymbol{y}(t)=\boldsymbol{h}[t,\boldsymbol{x}(t),\boldsymbol{u}(t)],\end{cases}\tag{3}$$

where $\boldsymbol{x}(t)\in\mathbb{R}^n$, $\boldsymbol{u}(t)\in\mathbb{R}^m$, $\boldsymbol{y}(t)\in\mathbb{R}^m$, the function $\boldsymbol{f}\in\{\boldsymbol{f}|\mathbb{R}\times\mathbb{C}\times\mathbb{R}^n\to\mathbb{R}^n\}$ and $\boldsymbol{h}\in\{\boldsymbol{h}|\mathbb{R}\times\mathbb{C}\times\mathbb{R}^m\to\mathbb{R}^m\}$ satisfying

$$\boldsymbol{f}(t,\boldsymbol{0},\boldsymbol{0})=\boldsymbol{0},\ \ \boldsymbol{h}(t,\boldsymbol{0},\boldsymbol{0})=\boldsymbol{0}.\tag{4}$$

Then the mapping from $\boldsymbol{u}$ to $\boldsymbol{y}$ can be written as $\boldsymbol{y}=\boldsymbol{G}\boldsymbol{u}$, and the mapping from $\boldsymbol{y}$ to $\boldsymbol{u}$ is $\boldsymbol{u}=\boldsymbol{\Delta}\boldsymbol{y}$, if there exists a nonsingular constant matrix $\boldsymbol{X}$ such that

$$\gamma(\boldsymbol{G}_X)=\|\boldsymbol{X}^{-1}\boldsymbol{G}\boldsymbol{X}\|_\infty<1,\ \gamma(\boldsymbol{\Delta}_X)=\|\boldsymbol{X}^{-1}\boldsymbol{\Delta}\boldsymbol{X}\|_\infty\le1,\tag{5}$$

and the system (3) is regular and impulse-free. Then the singular system (3) is said to be input-output stable.

We conclude this section by presenting the following lemma which will be used in the proof of our main results in the following section. This lemma is a simple variant of Lemma 1 in (Xu *et al*., 2003a).

Lemma 1 The following singular system

$$\begin{cases} \boldsymbol{E}\dot{\boldsymbol{x}}(t)=\boldsymbol{A}\boldsymbol{x}(t)+\boldsymbol{B}\boldsymbol{u}(t), \\ \boldsymbol{y}(t)=\boldsymbol{C}\boldsymbol{x}(t)+\boldsymbol{D}\boldsymbol{u}(t), \end{cases} \tag{6}$$

is regular, impulse-free, stable and satisfies

$$\gamma[\boldsymbol{G}(s)]=\|\boldsymbol{G}(s)\|_\infty=\|\boldsymbol{C}(s\boldsymbol{E}-\boldsymbol{A})^{-1}\boldsymbol{B}+\boldsymbol{D}\|_\infty<1,$$

if and only if there exist a symmetric positive definite matrix $\boldsymbol{P}$, matrices $\boldsymbol{S}$ and $\boldsymbol{S}_1$ with appropriate dimensions such that

$$\begin{bmatrix} (1,1) & (1,2) \\ (1,2)^{\mathrm{T}} & (2,2) \end{bmatrix}<\boldsymbol{0}, \tag{7}$$

where

$$(1,1)=(\boldsymbol{E}\boldsymbol{P}+\boldsymbol{S}\boldsymbol{R}^{\mathrm{T}})\boldsymbol{A}^{\mathrm{T}}+\boldsymbol{A}(\boldsymbol{P}\boldsymbol{E}^{\mathrm{T}}+\boldsymbol{R}\boldsymbol{S}^{\mathrm{T}})+\boldsymbol{B}\boldsymbol{B}^{\mathrm{T}},$$
$$(1,2)=(\boldsymbol{E}\boldsymbol{P}+\boldsymbol{S}\boldsymbol{R}^{\mathrm{T}})\boldsymbol{C}^{\mathrm{T}}+\boldsymbol{A}\boldsymbol{R}\boldsymbol{S}_1^{\mathrm{T}}+\boldsymbol{B}\boldsymbol{D}^{\mathrm{T}},$$
$$(2,2)=-(\boldsymbol{I}-\boldsymbol{S}_1\boldsymbol{R}^{\mathrm{T}}\boldsymbol{C}^{\mathrm{T}}-\boldsymbol{C}\boldsymbol{R}\boldsymbol{S}_1^{\mathrm{T}}-\boldsymbol{D}\boldsymbol{D}^{\mathrm{T}}),$$

$\boldsymbol{R}\in\mathbb{R}^{n\times(n-r)}$ is any matrix with full column rank and satisfies $\boldsymbol{E}\boldsymbol{R}=\boldsymbol{0}$.

MAIN RESULTS

The nominal unforced singular system of system (1) can be written as:

$$\boldsymbol{E}\dot{\boldsymbol{x}}(t)=\boldsymbol{A}\boldsymbol{x}(t)+\boldsymbol{A}_\tau\boldsymbol{x}(t-\tau(t))+\int_{-h(t)}^{0}\boldsymbol{A}_h\boldsymbol{x}(t+\theta)\mathrm{d}\theta. \tag{8}$$

We first discuss the robust stability of the nominal singular system (8), which will play a key role in obtaining the results of robust stability and stabilization for uncertain singular system (1). To this end, we introduce a new input-output model.

Following the same philosophy as that in (Gu *et al*., 2003), by letting

$$\begin{cases} \boldsymbol{y}_1(t)=\boldsymbol{x}(t),\ \ \boldsymbol{u}_1(t)=\boldsymbol{y}_1(t-\tau(t)), \\ \boldsymbol{y}_2(t)=\boldsymbol{A}_h\boldsymbol{x}(t),\ \ \boldsymbol{u}_2(t)=\dfrac{1}{h}\displaystyle\int_{-h(t)}^{0}\boldsymbol{y}_2(t+\theta)\mathrm{d}\theta, \end{cases} \tag{9}$$

the system (8) reduces to

$$\begin{cases} \boldsymbol{E}\dot{\boldsymbol{x}}(t)=\tilde{\boldsymbol{A}}\boldsymbol{x}(t)+\tilde{\boldsymbol{B}}\boldsymbol{u}(t), \\ \boldsymbol{y}(t)=\tilde{\boldsymbol{C}}\boldsymbol{x}(t)+\tilde{\boldsymbol{D}}\boldsymbol{u}(t), \end{cases} \tag{10}$$

where

$$\tilde{\boldsymbol{A}}=\boldsymbol{A},\ \tilde{\boldsymbol{B}}=[\boldsymbol{A}_\tau\ \ h\boldsymbol{I}],\ \tilde{\boldsymbol{C}}=[\boldsymbol{I}_n\ \ \boldsymbol{A}_h^{\mathrm{T}}]^{\mathrm{T}},\ \tilde{\boldsymbol{D}}=\boldsymbol{0},$$
$$\boldsymbol{u}=[\boldsymbol{u}_1^{\mathrm{T}}(t)\ \ \boldsymbol{u}_2^{\mathrm{T}}(t)]^{\mathrm{T}},\ \ \boldsymbol{y}=[\boldsymbol{y}_1^{\mathrm{T}}(t)\ \ \boldsymbol{y}_2^{\mathrm{T}}(t)]^{\mathrm{T}}.$$

Therefore, the robust stability of system (8) is equivalent to the input-output stability of system (10). For the system (10), we first discuss the stability from output to input, that is, to verify whether $\gamma(\varDelta_X)=\|\boldsymbol{X}^{-1}\boldsymbol{\varDelta X}\|_\infty\le1$ is satisfied.

Let $\mathscr{R}$ be a family of real nonsingular matrices, that is $\mathscr{R}=\{\mathrm{diag}(\boldsymbol{X}_1,\boldsymbol{X}_2)\mid\boldsymbol{X}_1,\boldsymbol{X}_2\in\mathbb{R}^{n\times n}\}$, where $\boldsymbol{X}_1$, $\boldsymbol{X}_2$ are nonsingular matrices. For any nonsingular matrix $\boldsymbol{X}\in\mathscr{R}$, we define $\boldsymbol{Z}=\mathrm{diag}(\boldsymbol{Z}_1,\boldsymbol{Z}_2)=\boldsymbol{X}^{\mathrm{T}}\boldsymbol{X}$, and let $\mathfrak{F}$ be the class of positive definite block-diagonal matrices, that is, $\mathfrak{F}=\{\boldsymbol{Z}=\mathrm{diag}(\boldsymbol{Z}_1,\boldsymbol{Z}_2)>\boldsymbol{0}\}|\boldsymbol{Z}_1,\boldsymbol{Z}_2\in\mathbb{R}^{n\times n}\}$. Clearly, $\mathfrak{F}=\{\boldsymbol{X}^{\mathrm{T}}\boldsymbol{X}|\boldsymbol{X}\in\mathscr{R}\}$.

Let operator $\varDelta=\mathrm{diag}(\varDelta_1,\varDelta_2)$, that is $\boldsymbol{u}_1=\varDelta_1\boldsymbol{y}_1$, $\boldsymbol{u}_2=\varDelta_2\boldsymbol{y}_2$, for any $t>0$, $\boldsymbol{Z}\in\mathfrak{F}$, if the following integral quadratic constraint

$$\int_0^t\boldsymbol{u}^{\mathrm{T}}(\eta)\boldsymbol{Z}\boldsymbol{u}(\eta)\mathrm{d}\eta\le\int_0^t\boldsymbol{y}^{\mathrm{T}}(\eta)\boldsymbol{Z}\boldsymbol{y}(\eta)\mathrm{d}\eta, \tag{11}$$
$$\boldsymbol{u}=[\boldsymbol{u}_1^{\mathrm{T}}(t)\ \ \boldsymbol{u}_2^{\mathrm{T}}(t)]^{\mathrm{T}},\ \boldsymbol{y}=[\boldsymbol{y}_1^{\mathrm{T}}(t)\ \ \boldsymbol{y}_2^{\mathrm{T}}(t)]^{\mathrm{T}}$$

holds, then for any $\boldsymbol{X}\in\mathscr{R}$, we have

$$\begin{aligned}&\int_0^t\|\varDelta_X\boldsymbol{y}(\eta)\|^2\mathrm{d}\eta-\int_0^t\|\boldsymbol{y}(\eta)\|^2\,\mathrm{d}\eta\\&=\int_0^t(\varDelta\tilde{\boldsymbol{y}}(\eta))^{\mathrm{T}}\boldsymbol{Z}\varDelta\tilde{\boldsymbol{y}}(\eta)\mathrm{d}\eta-\int_0^t\tilde{\boldsymbol{y}}^{\mathrm{T}}(\eta)\boldsymbol{Z}\tilde{\boldsymbol{y}}(\eta)\mathrm{d}\eta\\&\le0,\end{aligned} \tag{12}$$

where $\tilde{\boldsymbol{y}}=\boldsymbol{X}^{-1}\boldsymbol{y}$, which implies that $\gamma(\varDelta_X)\le1$.

Therefore, if we can testify inequality (11), the system (10) is stable from output to input. So, using zero initial conditions, we have

$$\begin{aligned}\int_0^t\boldsymbol{u}_1^{\mathrm{T}}(\eta)\boldsymbol{Z}_1\boldsymbol{u}_1(\eta)\mathrm{d}\eta&=\int_{-\tau(t)}^{t-\tau(t)}\boldsymbol{y}_1^{\mathrm{T}}(\eta)\boldsymbol{Z}_1\boldsymbol{y}_1(\eta)\mathrm{d}\eta\\&\le\int_0^t\boldsymbol{y}_1^{\mathrm{T}}(\eta)\boldsymbol{Z}_1\boldsymbol{y}_1(\eta)\mathrm{d}\eta.\end{aligned} \tag{13}$$

Also, using Jensen's Inequality (Gu *et al*., 2003) and exchanging the order of integration, we can obtain

$$\int_0^t \boldsymbol{u}_2^{\mathrm{T}}(\eta)\boldsymbol{Z}_2\boldsymbol{u}_2(\eta)\mathrm{d}\eta$$
$$=\frac{1}{h^2}\int_0^t\left[\int_{-h(t)}^0 \boldsymbol{y}_2(\eta+\theta)\mathrm{d}\theta\right]^{\mathrm{T}}\boldsymbol{Z}_2\left[\int_{-h(t)}^0 \boldsymbol{y}_2(\eta+\theta)\mathrm{d}\theta\right]\mathrm{d}\eta$$
$$\le\int_0^t \boldsymbol{y}_2^{\mathrm{T}}(\eta)\boldsymbol{Z}_2\boldsymbol{y}_2(\eta)\mathrm{d}\eta. \tag{14}$$

Then inequalities (13) and (14) imply inequality (11) holds, and thus implies the system (10) is stable from output to input.

Based on Lemma 1 and the discussion above, the following theorem gives the solution to the robust stability for the nominal singular system (8).

Theorem 1 The nominal singular system (8) is robustly stable if and only if there exist symmetric positive definite matrices $\boldsymbol{P}$, $\boldsymbol{L}_1$, $\boldsymbol{L}_2$, matrices $\boldsymbol{S}$, $\boldsymbol{Y}_1$ and $\boldsymbol{Y}_2$ with appropriate dimensions such that

$$\begin{bmatrix} (1,1) & (1,2) & (1,3) \\ (1,2)^{\mathrm{T}} & (2,2) & (2,3) \\ (1,3)^{\mathrm{T}} & (2,3)^{\mathrm{T}} & (3,3) \end{bmatrix} < \boldsymbol{0}, \tag{15}$$

where

$$(1,1)=\boldsymbol{A}(\boldsymbol{P}\boldsymbol{E}^{\mathrm{T}}+\boldsymbol{R}\boldsymbol{S}^{\mathrm{T}})+(\boldsymbol{E}\boldsymbol{P}+\boldsymbol{S}\boldsymbol{R}^{\mathrm{T}})\boldsymbol{A}^{\mathrm{T}}+\boldsymbol{h}^2\boldsymbol{L}_2+\boldsymbol{A}_\tau\boldsymbol{L}_1\boldsymbol{A}_\tau^{\mathrm{T}},$$
$$(1,2)=\boldsymbol{E}\boldsymbol{P}+\boldsymbol{S}\boldsymbol{R}^{\mathrm{T}}+\boldsymbol{A}\boldsymbol{R}\boldsymbol{Y}_1^{\mathrm{T}},$$
$$(1,3)=(\boldsymbol{E}\boldsymbol{P}+\boldsymbol{S}\boldsymbol{R}^{\mathrm{T}})\boldsymbol{A}_h^{\mathrm{T}}+\boldsymbol{A}\boldsymbol{R}\boldsymbol{Y}_2^{\mathrm{T}},$$
$$(2,2)=-\boldsymbol{L}_1+\boldsymbol{Y}_1\boldsymbol{R}^{\mathrm{T}}+\boldsymbol{R}\boldsymbol{Y}_1^{\mathrm{T}},\ \ (2,3)=\boldsymbol{R}\boldsymbol{Y}_2^{\mathrm{T}}+\boldsymbol{Y}_1\boldsymbol{R}^{\mathrm{T}}\boldsymbol{A}_h^{\mathrm{T}},$$
$$(3,3)=-\boldsymbol{L}_2+\boldsymbol{A}_h\boldsymbol{R}\boldsymbol{Y}_2^{\mathrm{T}}+\boldsymbol{Y}_2\boldsymbol{R}^{\mathrm{T}}\boldsymbol{A}_h^{\mathrm{T}},$$

$\boldsymbol{R}\in\mathbb{R}^{n\times(n-r)}$ is any matrix with full column rank and satisfies $\boldsymbol{E}\boldsymbol{R}=\boldsymbol{0}$.

Proof From the discussion above, the singular system (10) is stable from output to input, that is, $\gamma(\boldsymbol{\Delta}_X)\le 1$. Therefore, if we can testify $\gamma(\boldsymbol{G}_X)<1$ and the singular system (10) is regular, impulse-free, according to Definition 3, the singular system (10) is input-output stable, which implies that the nominal singular system (8) is robustly stable. To this end, first according to the definition of norm, we have

$$\gamma(\boldsymbol{G}_X)=\|\boldsymbol{X}\boldsymbol{G}(s)\boldsymbol{X}^{-1}\|_\infty=\|\boldsymbol{G}(s)\|_\infty=\gamma(\boldsymbol{G}). \tag{16}$$

On the other hand, according to the definition of transfer function, for the system (10), we have

$$\gamma(\boldsymbol{G}_X)=\|\boldsymbol{X}[\tilde{\boldsymbol{C}}(s\boldsymbol{E}-\tilde{\boldsymbol{A}})^{-1}\tilde{\boldsymbol{B}}+\tilde{\boldsymbol{D}}]\boldsymbol{X}^{-1}\|_\infty. \tag{17}$$

Therefore, according to Lemma 1, the singular system (10) is regular, impulse-free and $\gamma(\boldsymbol{G}_X)<1$ if and only if there exist a symmetric positive definite matrix $\boldsymbol{P}$, matrices $\boldsymbol{S}$ and $\boldsymbol{S}_1$ with appropriate dimension such that

$$\begin{bmatrix} (1,1) & (1,2) \\ (1,2)^{\mathrm{T}} & (2,2) \end{bmatrix} < \boldsymbol{0}, \tag{18}$$

where

$$(1,1)=(\boldsymbol{E}\boldsymbol{P}+\boldsymbol{S}\boldsymbol{R}^{\mathrm{T}})\tilde{\boldsymbol{A}}^{\mathrm{T}}+\tilde{\boldsymbol{A}}(\boldsymbol{P}\boldsymbol{E}^{\mathrm{T}}+\boldsymbol{R}\boldsymbol{S}^{\mathrm{T}})+\tilde{\boldsymbol{B}}\boldsymbol{X}^{-1}\boldsymbol{X}^{-\mathrm{T}}\tilde{\boldsymbol{B}}^{\mathrm{T}},$$
$$(1,2)=(\boldsymbol{E}\boldsymbol{P}+\boldsymbol{S}\boldsymbol{R}^{\mathrm{T}})\tilde{\boldsymbol{C}}^{\mathrm{T}}\boldsymbol{X}^{\mathrm{T}}+\tilde{\boldsymbol{A}}\boldsymbol{R}\boldsymbol{S}_1^{\mathrm{T}}+\tilde{\boldsymbol{B}}\boldsymbol{X}^{-1}\boldsymbol{X}^{-\mathrm{T}}\tilde{\boldsymbol{D}}^{\mathrm{T}}\boldsymbol{X}^{\mathrm{T}},$$
$$(2,2)=-\boldsymbol{I}+\boldsymbol{S}_1\boldsymbol{R}^{\mathrm{T}}\tilde{\boldsymbol{C}}^{\mathrm{T}}\boldsymbol{X}^{\mathrm{T}}+\boldsymbol{X}\tilde{\boldsymbol{C}}\boldsymbol{R}\boldsymbol{S}_1^{\mathrm{T}}+\boldsymbol{X}\tilde{\boldsymbol{D}}\boldsymbol{X}^{-1}\boldsymbol{X}^{-\mathrm{T}}\tilde{\boldsymbol{D}}^{\mathrm{T}}\boldsymbol{X}^{\mathrm{T}}.$$

Pre-multiplying diag($\boldsymbol{I}$, $\boldsymbol{X}^{-1}$) and post-multiplying diag($\boldsymbol{I}$, $\boldsymbol{X}^{-\mathrm{T}}$) to the left and right sides of inequality (18), and letting

$$\boldsymbol{L}=\begin{bmatrix} \boldsymbol{L}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{L}_2 \end{bmatrix}=(\boldsymbol{X}^{\mathrm{T}}\boldsymbol{X})^{-1},\ \ \boldsymbol{X}^{-1}\boldsymbol{S}_1=\begin{bmatrix} \boldsymbol{Y}_1 \\ \boldsymbol{Y}_2 \end{bmatrix}, \tag{19}$$

then inequality (18) can be transformed into inequality (15).

Therefore, if inequality (15) is satisfied, inequality (18) is also satisfied, that is the singular system (10) is regular, impulse-free and $\gamma(\boldsymbol{G}_X)<1$. On the other hand, if the singular system (10) is regular, impulse-free and $\gamma(\boldsymbol{G}_X)<1$, inequality (18) holds, and then inequality (15) holds. According to Definition 3, the singular system (10) is input-output stable.

Since the input-output stability of singular system (10) is equivalent to the robust stability of nominal singular system (8), the proof is completed.

We introduce the following lemma, which will be used in designing robust state feedback control law.

Lemma 2 (Petersen, 1987) Given matrices $\boldsymbol{\Gamma}$, $\boldsymbol{\Lambda}$ and symmetric matrix $\boldsymbol{\Omega}$, we have $\boldsymbol{\Omega}+\boldsymbol{\Gamma}\boldsymbol{F}\boldsymbol{\Lambda}+\boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{F}^{\mathrm{T}}\boldsymbol{\Gamma}^{\mathrm{T}}<\boldsymbol{0}$ for any $\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}\le\boldsymbol{I}$, if and only if there exists a scalar $\varepsilon>0$ such that

$$\boldsymbol{\Omega}+\varepsilon^{-1}\boldsymbol{\Gamma}\boldsymbol{\Gamma}^{\mathrm{T}}+\varepsilon\boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{\Lambda}<\boldsymbol{0}.$$

As for the robust stabilization problem of the uncertain singular system (1), we can obtain the following theorem according to Theorem 1:

Theorem 2 Consider the uncertain singular system (1), if and only if there exist symmetric positive definite matrices $\boldsymbol{P}$, $\boldsymbol{L}_1$, $\boldsymbol{L}_2$, matrices $\boldsymbol{S}$, $\boldsymbol{W}$, $\boldsymbol{Y}_1$, $\boldsymbol{Y}_2$ with appropriate dimensions, scalars $\varepsilon_1>0$ and $\varepsilon_2>0$ such that

$$\begin{bmatrix} (1,1) & (1,2) & (1,3) & \boldsymbol{A}_\tau \boldsymbol{L}_1 & (1,5) & (1,6) \\ * & (2,2) & (2,3) & \boldsymbol{0} & (2,5) & (2,6) \\ * & * & (3,3) & \boldsymbol{0} & (3,5) & (3,6) \\ * & * & * & -\boldsymbol{L}_1 & \boldsymbol{L}_1\boldsymbol{N}_\tau^{\mathrm{T}} & \boldsymbol{0} \\ * & * & * & * & -\varepsilon_1 \boldsymbol{I} & \boldsymbol{0} \\ * & * & * & * & * & -\varepsilon_2 \boldsymbol{I} \end{bmatrix} < \boldsymbol{0}, \quad (20)$$

where

$$(1,1) = \boldsymbol{A}(\boldsymbol{PE}+\boldsymbol{RS}^{\mathrm{T}}) + (\boldsymbol{EP}+\boldsymbol{SR}^{\mathrm{T}})\boldsymbol{A}^{\mathrm{T}} + \boldsymbol{BW} + \boldsymbol{W}^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}} + h^2\boldsymbol{L}_2 + \varepsilon_1 \boldsymbol{MM}^{\mathrm{T}},$$
$$(1,2) = \boldsymbol{EP}+\boldsymbol{SR}^{\mathrm{T}} + \boldsymbol{ARY}_1^{\mathrm{T}} + \boldsymbol{BW}(\boldsymbol{PE}^{\mathrm{T}}+\boldsymbol{RS}^{\mathrm{T}})^{-1}\boldsymbol{RY}_1^{\mathrm{T}},$$
$$(1,3) = (\boldsymbol{EP}+\boldsymbol{SR}^{\mathrm{T}})\boldsymbol{A}_h^{\mathrm{T}} + \boldsymbol{ARY}_2^{\mathrm{T}} + \boldsymbol{BW}(\boldsymbol{PE}^{\mathrm{T}}+\boldsymbol{RS}^{\mathrm{T}})^{-1}\boldsymbol{RY}_2^{\mathrm{T}},$$
$$(1,5) = (\boldsymbol{EP}+\boldsymbol{SR}^{\mathrm{T}})\boldsymbol{N}_a^{\mathrm{T}} + \boldsymbol{W}^{\mathrm{T}}\boldsymbol{N}_b^{\mathrm{T}},$$
$$(1,6) = (\boldsymbol{EP}+\boldsymbol{SR}^{\mathrm{T}})\boldsymbol{N}_h^{\mathrm{T}},$$
$$(2,2) = -\boldsymbol{L}_1 + \boldsymbol{Y}_1\boldsymbol{R}^{\mathrm{T}} + \boldsymbol{RY}_1^{\mathrm{T}}, \quad (2,3) = \boldsymbol{RY}_2^{\mathrm{T}} + \boldsymbol{Y}_1\boldsymbol{R}^{\mathrm{T}}\boldsymbol{A}_h^{\mathrm{T}},$$
$$(2,5) = \boldsymbol{Y}_1\boldsymbol{R}^{\mathrm{T}}\boldsymbol{N}_a^{\mathrm{T}} + \boldsymbol{Y}_1\boldsymbol{R}^{\mathrm{T}}(\boldsymbol{EP}+\boldsymbol{SR}^{\mathrm{T}})^{-1}\boldsymbol{W}^{\mathrm{T}}\boldsymbol{N}_b^{\mathrm{T}},$$
$$(2,6) = \boldsymbol{Y}_1\boldsymbol{R}^{\mathrm{T}}\boldsymbol{N}_h^{\mathrm{T}},$$
$$(3,3) = -\boldsymbol{L}_2 + \boldsymbol{A}_h\boldsymbol{RY}_2^{\mathrm{T}} + \boldsymbol{Y}_2\boldsymbol{R}^{\mathrm{T}}\boldsymbol{A}_h^{\mathrm{T}} + \varepsilon_2\boldsymbol{MM}^{\mathrm{T}},$$
$$(3,5) = \boldsymbol{Y}_2\boldsymbol{R}^{\mathrm{T}}\boldsymbol{N}_a^{\mathrm{T}} + \boldsymbol{Y}_2\boldsymbol{R}^{\mathrm{T}}(\boldsymbol{EP}+\boldsymbol{SR}^{\mathrm{T}})^{-1}\boldsymbol{W}^{\mathrm{T}}\boldsymbol{N}_b^{\mathrm{T}},$$
$$(3,6) = \boldsymbol{Y}_2\boldsymbol{R}^{\mathrm{T}}\boldsymbol{N}_h^{\mathrm{T}}.$$

Then, we can construct a state feedback control law

$$\boldsymbol{u}(t) = \boldsymbol{W}(\boldsymbol{PE}+\boldsymbol{RS}^{\mathrm{T}})^{-1}\boldsymbol{x}(t), \quad (21)$$

such that the resultant closed-loop system is robustly stable for all admissible uncertainties, where $\boldsymbol{R}\in\mathbb{R}^{n\times(n-r)}$ is any matrix with full column rank and satisfies $\boldsymbol{ER}=\boldsymbol{0}$.

Proof We can assume that $\boldsymbol{Q}=\boldsymbol{EP}+\boldsymbol{SR}^{\mathrm{T}}$ is invertible, otherwise we can choose a sufficiently small scalar $\alpha>0$ such that $\hat{\boldsymbol{Q}} = \boldsymbol{Q}+\alpha\boldsymbol{I}$ is also invertible.

Then, replacing $\boldsymbol{A}$ and $\boldsymbol{N}_a$ by $\boldsymbol{A}+\boldsymbol{BK}$ and $\boldsymbol{N}_a+\boldsymbol{N}_b\boldsymbol{K}$, respectively, and following the similar line as that in the proof of Theorem 1 and using Lemma 2, inequality (20) follows immediately for $\boldsymbol{K}=\boldsymbol{W}(\boldsymbol{PE}+\boldsymbol{RS}^{\mathrm{T}})^{-1}$.

Inequality (20) is nonlinear matrix inequality, which is difficult to implement numerically. Since $\boldsymbol{S}_1$ is any matrix, supposing $\boldsymbol{S}_1=\boldsymbol{0}$, from $\boldsymbol{X}^{-1}\boldsymbol{S}_1=[\boldsymbol{Y}_1^{\mathrm{T}}\ \boldsymbol{Y}_2^{\mathrm{T}}]^{\mathrm{T}}$, then $\boldsymbol{Y}_1=\boldsymbol{0}$ and $\boldsymbol{Y}_2=\boldsymbol{0}$, we can deduce the following more conservative result based on strict LMI:

Corollary 1 Consider the uncertain singular system (1), if and only if there exist symmetric positive definite matrices $\boldsymbol{P}$, $\boldsymbol{L}_1$, $\boldsymbol{L}_2$, matrices $\boldsymbol{S}$, $\boldsymbol{W}$ with appropriate dimensions, scalars $\varepsilon_1>0$ and $\varepsilon_2>0$ such that

$$\begin{bmatrix} (1,1) & (1,2) & (1,3) & \boldsymbol{A}_\tau \boldsymbol{L}_1 & (1,5) & (1,6) \\ * & -\boldsymbol{L}_1 & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ * & * & (3,3) & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ * & * & * & -\boldsymbol{L}_1 & \boldsymbol{L}_1\boldsymbol{N}_\tau^{\mathrm{T}} & \boldsymbol{0} \\ * & * & * & * & -\varepsilon_1 \boldsymbol{I} & \boldsymbol{0} \\ * & * & * & * & * & -\varepsilon_2 \boldsymbol{I} \end{bmatrix} < \boldsymbol{0}, \quad (22)$$

where

$$(1,1) = \boldsymbol{A}(\boldsymbol{PE}+\boldsymbol{RS}^{\mathrm{T}}) + (\boldsymbol{EP}+\boldsymbol{SR}^{\mathrm{T}})\boldsymbol{A}^{\mathrm{T}} + \boldsymbol{BW} + \boldsymbol{W}^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}} + h^2\boldsymbol{L}_2 + \varepsilon_1 \boldsymbol{MM}^{\mathrm{T}},$$
$$(1,2) = \boldsymbol{EP}+\boldsymbol{SR}^{\mathrm{T}}, \quad (1,3) = (\boldsymbol{EP}+\boldsymbol{SR}^{\mathrm{T}})\boldsymbol{A}_h^{\mathrm{T}},$$
$$(1,5) = (\boldsymbol{EP}+\boldsymbol{SR}^{\mathrm{T}})\boldsymbol{N}_a^{\mathrm{T}} + \boldsymbol{W}^{\mathrm{T}}\boldsymbol{N}_b^{\mathrm{T}},$$
$$(1,6) = (\boldsymbol{EP}+\boldsymbol{SR}^{\mathrm{T}})\boldsymbol{N}_h^{\mathrm{T}}, \quad (3,3) = -\boldsymbol{L}_2 + \varepsilon_2\boldsymbol{MM}^{\mathrm{T}}.$$

Then, we can construct a state feedback control law

$$\boldsymbol{u}(t) = \boldsymbol{W}(\boldsymbol{PE}+\boldsymbol{RS}^{\mathrm{T}})^{-1}\boldsymbol{x}(t), \quad (23)$$

such that the resultant closed-loop system is robustly stable for all admissible uncertainties, where $\boldsymbol{R}\in\mathbb{R}^{n\times(n-r)}$ is any matrix with full column rank and satisfies $\boldsymbol{ER}=\boldsymbol{0}$.

NUMERICAL SIMULATION

In this section, we give an example to demonstrate the effectiveness of the proposed method.

Consider the uncertain singular system (1) with parameters as follows:

$$\boldsymbol{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \boldsymbol{A} = \begin{bmatrix} 1.5 & 0.5 & 1 \\ -1 & 0 & 1 \\ 0.5 & 0 & 1 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 1 & 5 \\ 0.5 & 0.3 \\ 10 & 2 \end{bmatrix},$$
$$\boldsymbol{A}_\tau = \begin{bmatrix} -1 & 0 & -1 \\ 1 & -1 & -0.5 \\ 0.3 & 0.5 & -1 \end{bmatrix}, \boldsymbol{A}_h = \begin{bmatrix} 1 & 5 & 1 \\ 2 & -0.5 & 0.3 \\ -1 & 1 & 0.2 \end{bmatrix}, \boldsymbol{M} = \begin{bmatrix} 0.5 \\ 0.2 \\ 0.1 \end{bmatrix},$$
$$\boldsymbol{N}_\tau = [0.1\ \ 0.2\ \ 0.5],\ \boldsymbol{N}_h = [0.1\ \ 0.1\ \ 0.1],$$
$$\boldsymbol{N}_a = [0.2\ \ 0.1\ \ 0.3],\ \boldsymbol{N}_b = [0.1\ \ 0.1].$$

In this example, we assume that h=3.5 and the uncertain matrix $\boldsymbol{F}(t)$=sin t. The purpose is to design a state feedback control such that the resultant closed-loop system is robustly stable for all admissible uncertainties. To this end, we choose $\boldsymbol{R}$=[0 0 1]$^{\mathrm{T}}$. According to Corollary 1, using Matlab LMI Control Toolbox to solve the LMI (22), we obtain the solution as follows:

$$\boldsymbol{P}=\begin{bmatrix}0.7107 & 0.6003 & 0\\ 0.6003 & 0.6831 & 0\\ 0 & 0 & 4.5397\end{bmatrix},\ \boldsymbol{S}=\begin{bmatrix}-3.1267\\ -2.9535\\ 0.1437\end{bmatrix},$$

$$\boldsymbol{W}=\begin{bmatrix}3.1331 & -0.0868 & -6.6544\\ -14.0812 & -0.3137 & 2.7735\end{bmatrix},$$

$$\boldsymbol{L}_1=\begin{bmatrix}2.1162 & 1.4466 & -0.3035\\ 1.4466 & 2.3078 & -1.0823\\ -0.3035 & -1.0823 & 3.9739\end{bmatrix},$$

$$\boldsymbol{L}_2=\begin{bmatrix}4.8196 & 0.1916 & -0.5117\\ 0.1916 & 0.0722 & 0.1805\\ -0.5117 & 0.1805 & 4.4920\end{bmatrix},$$

$$\varepsilon_1=5.9988,\ \ \varepsilon_2=6.9979.$$

Therefore, a robustly stabilizing state feedback control can be obtained as

$$\boldsymbol{u}(t)=\begin{bmatrix}-116.7909 & -97.7612 & -46.3192\\ -19.3850 & 100.0480 & 19.3056\end{bmatrix}\boldsymbol{x}(t).$$

RESULTS AND DISCUSSION

The problems of robust stability and stabilization for uncertain singular systems with time-varying discrete and distributed delays are studied. Based on a new input-output model, some sufficient and necessary conditions are presented by strict LMIs. The explicit expression of the robust state feedback control law, which guarantees the resultant closed-loop system robustly stable for all admissible uncertainties, is also given. Finally, a numerical example is presented to demonstrate the effectiveness of the proposed method.

References

Bliman, P.A., 2000. Extension of Popov absolute stability criterion to non-autonomous systems with delays. *Int. J. Control*, **73**:1349-1361. [doi:10.1080/002071700445370]

Cao, Y., Sun, Y., Lam, J., 1998. Delay dependent robust H_∞ control for uncertain systems with time-varying delays. *IEE Proc.-Control Theory & Appl.*, **145**:338-344. [doi:10.1049/ip-cta:19981951]

Dai, L., 1989. Singular Control Systems. Springer-Verlag, Berlin, Germany.

Desoer, C.A., Vidyasagar, M., 1975. Feedback Systems: Input-Output Properties. Academic Press, New York.

Fridman, E., Shaked, U., 2006. Input-output approach to stability and L_1-gain analysis of systems with time-varying delays. *Syst. & Control Lett.*, **55**:1041-1053. [doi:10.1016/j.sysconle.2006.07.002]

Gu, K., Kharitonov, V.L., Chen, J., 2003. Stability of Time-Delay Systems. Birkhauser, Boston, p.275-323.

Huang, Y.P., Zhou, K., 2000. Robust stability of uncertain time-delay systems. *IEEE Trans. on Automatic Control*, **45**:2169-2173. [doi:10.1109/9.887666]

Lewis, F.L., 1986. A survey of linear singular systems. *Circuits, Systems & Signal Processing*, **5**:3-36. [doi:10.1007/BF01600184]

Ma, B., Zhu, X., Sun, J., 2005. Guaranteed cost control of uncertain T-S fuzzy descriptor systems. Proc. Int. Conf. on Machine Learning and Cybernetics, **2**:742-746. [doi:10.1109/ICMLC.2005.1527042]

Mahmoud, M., 2000. Robust Control and Filtering for Time-Delay Systems. Marcel Decker, New York.

Masubuchi, I., Kamitane, Y., Ohara, A., Suda, N., 1997. H_∞ control for descriptor systems: a matrix inequality approach. *Automatica*, **33**:669-673. [doi:10.1016/S0005-1098(96)00193-8]

Niculoscu, S.I., 2001. Delay Effects on Stability—A Robust Control Approach. Springer-Verlag, London.

Petersen, I.R., 1987. A stabilization algorithm for a class of uncertain linear systems. *Syst. & Control Lett.*, **8**:351-357. [doi:10.1016/0167-6911(87)90102-2]

Vidyasagar, M., 1993. Nonlinear Systems Analysis (2nd Ed.). Prentice-Hall, Englewood Cliffs, NJ.

Wang, H., Xue, A., Lu, R., Xu, Z., Wang, J., 2007. Robust H_∞ control for discrete singular systems with parameter uncertainties. *Acta Automatica Sinica*, **33**:1300-1305 (in Chinese). [doi:10.1360/aas-007-1300]

Xu, S., Dooren, P.V., Stefan, R., Lam, J., 2002. Robust stability and stabilization for singular systems with state delay and parameter uncertainty. *IEEE Trans. on Automatic Control*, **47**:1122-1128. [doi:10.1109/TAC.2002.800651]

Xu, S., Lam, J., Liu, W., Zhang, Q., 2003a. H_∞ model reduction for singular systems: continuous-time case. *IEE Proc.-Control Theory & Appl.*, **150**:637-641. [doi:10.1049/ip-cta:20030815]

Xu, S., Lam, J., Yang, C., 2003b. Robust H_∞ control for uncertain singular systems with state delay. *Int. J. Rob. Nonl. Control*, **13**:1213-1223. [doi:10.1002/rnc.837]

Yue, D., Han, Q.L., 2005. Delay-dependent robust H_∞ controller design for uncertain descriptor systems with time-varying discrete and distributed delays. *IEE Proc.-Control Theory & Appl.*, **152**:628-638. [doi:10.1049/ip-cta:20045293]

Zhou, S., Lam, L., 2003. Robust stabilization of delayed singular systems with linear fractional parametric uncertainties. *Circuits, Systems & Signal Processing*, **22**:579-588. [doi:10.1007/s00034-003-1218-x]