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A numerical analysis to the non-linear fin problem

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Abstract: In this paper a numerical analysis is carried out to obtain the temperature distribution within a single fin. It is assumed that the heat transfer coefficient depends on the temperature. The complete highly non-linear problem is solved numerically and the variations of both, dimensionless surface temperature and dimensionless surface temperature gradient as well as heat transfer characteristics with the governing non-dimensional parameters of the problem are graphed and tabulated.

Key words: Fins, Ordinary differential equations (ODEs), Numerical solution, Heat transfer
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INTRODUCTION

Fins are commonly used to facilitate the dissipation of heat from a heated wall to the surrounding environment. The study of the non-linear fin problem has generated much interest in recent years in view of its numerous practical applications on semiconductors, heat exchangers, power generators and electronics components. The problem under investigation is highly non-linear and these classes of problems are not easy to examine. Therefore, many applied mathematicians and numerical analysts have also recently paid much attention in developing suitable algorithms for solving these problems. The highly non-linear heat-flow equation appeared from the non-linear fin problem leads to the need of a numerical treatment. This fact was pointed out by Hutcheon and Spalding (1958), who employed a resistance network analogue, could report that the solutions in the case of the heat flux is proportional to the local temperature to the power 5/4 under common conditions.

On the other hand, temperature-dependent functions for both heat transfer coefficient and thermal conductivity conduct to a lack of analytical solutions, which are scarce from the open literature. In view of all the above, well-known methods for non-linear problems, including perturbation method (Aziz, 1977), Adomian decomposition method (Chiu and Chen, 1977; Chang, 2005), Taylor series method (Kim and Huang, 2006; 2007; Kim *et al*., 2007) and variational iteration method (Tari *et al*., 2007), have been applied to the non-linear fin problem. Furthermore, Chowdhury and his co-workers (Chowdhury *et al*., 2007; Chowdhury and Hashim, 2007) presented very recently an application of the homotopy-perturbation method (HPM) to solve that problem. A systematic study of the effects of thermal radiation on the heat transfer from porous fins has been carried out by Kiwan (2007), and even attention has been paid to the transient analysis in (Yang *et al*., 2005).

Ganji *et al*.(2007) investigated the temperature distribution of a fin with variable thermal conductivity, whereas Domairry and Fazeli (2007) have very recently analyzed the fin efficiency for this problem. A systematic analysis of the application of He's variational iteration method to solve these heat-transfer problems has been carried out by Miansari *et al*.(2008) and Tari *et al*.(2007). Further, Ganji (2006) studied the application of HPM to these problems.

The present research endeavours to undertake a numerical study towards the non-linear fin problem by solving only one initial value problem (IVP) via 4th order Runge-Kutta algorithm together with shooting method. This numerical approach has already applied to fluid dynamics and related numerical studies are in (Cortell, 1995; 2005). The numerical results reported in (Cortell, 1995; 2005) have already been used for comparison (e.g., Taigbenu and Onyejekwe, 1999; Fang *et al*., 2006; Chang *et al*., 2006; Ishak *et al*., 2007a; 2007b). Usually, our proposed problems in the area of fluid dynamics are solved by using boundary-layer theory along with the concept of similarity solution (Cortell, 1994; 2005; 2006a; 2006b; 2007a; 2007b).

It is the purpose of this work to investigate numerically the effects of the governing parameters on temperature distribution occurring in a non-linear fin-type problem. The governing highly non-linear equation is solved using a Runge-Kutta shooting method and the numerical results for typically physical quantities of interest are obtained for different values of the convective-conductive parameter N. The cases of laminar or turbulent natural convection (i.e., m=5/4; m=4/3), nucleate boiling (i.e., m=3) and thermal radiation (i.e., m=4) are treated.

THEORY

Consider a straight fin of length l, cross-sectional area A and perimeter p. Its thermal conductivity is k, and the local heat transfer h along the fin surface is assumed to exhibit a power-law type dependence on the local temperature difference between the fin and the ambient fluid as:

$$h = C(T - T_{\mathrm{f}})^m, \tag{1}$$

where C is a constant derived from natural convection theory, T is the local temperature on the fin surface, T_{f} is the fluid temperature and the exponent m depends on the heat transfer mode. For more details, reader is referred to (Lesnic and Hegs, 2004; Liaw and Yeh, 1994).

The Fourier heat-conduction equation, combined with the condition that the state is steady, leads to the following differential equation:

$$Ak(\mathrm{d}^2 T / \mathrm{d} X^2) = pC(T - T_{\mathrm{f}})^m. \tag{2}$$

Define appropriately new dimensionless variables and quantities as:

$$\begin{aligned} &\theta = (T - T_{\mathrm{f}})/(T_{\mathrm{w}} - T_{\mathrm{f}}), \ x = X/l; \\ &N = pCl^2 (T_{\mathrm{w}} - T_{\mathrm{f}})^{m-1}/(Ak), \end{aligned} \tag{3}$$

where T_{w} is the fin base temperature. Eq.(2) becomes:

$$\mathrm{d}^2\theta/\mathrm{d}x^2 - N\theta^m = 0. \tag{4}$$

For simplicity, one can assume that the fin tip is insulated and the boundary conditions to Eq.(4) can be expressed as:

$$\theta'(0) = 0, \tag{5}$$

$$\theta(1) = 1. \tag{6}$$

The dimensionless coordinate x is measured along the fin length from its adiabatic end, and then we find that at the fin root (i.e., x=1) T=T_{w}, and consequently, $\theta(1)$=1. θ is the non-dimensional temperature, N is the convective-conductive parameter of the fin and the exponent m depends on the heat transfer mode. For practical interest the physical values of m are 5/4 and 4/3 for laminar natural convection and turbulent natural convection, respectively; 3 for nucleate boiling and 4 for thermal radiation.

NUMERICAL RESULTS

Eq.(4) can easily be written as the equivalent first-order system:

$$w_1' = w_2, \ w_2' = N w_1^m. \tag{7}$$

Here w_1=$\theta(x)$. In accordance with conditions Eqs.(5) and (6) we obtain

$$w_2(0) = 0, \ w_1(1) = 1. \tag{8}$$

Using numerical methods of integration and disregarding temporarily the second boundary condition Eq.(8), a family of solutions of Eq.(7) can be obtained for arbitrarily chosen values of $w_1(0)=(\theta(x))_{x=0}>0$. We can tentatively assume that a special value of $w_1(0)$ yields a solution for which the second condition Eq.(8) is met exactly.

We guess $w_1(0)$ and integrate Eq.(7) along with boundary conditions Eq.(8) as an initial value problem by the fourth-order Runge-Kutta method for the solution of high-order general initial value problems (Cortell, 1993).

Users do not like it when a program for solving the aforementioned IVP for a system of ordinary differential equations returns an approximation to a quantity like $\theta(1)$ different from unity (Eq.(6)). Unfortunately, no standard numerical method provides this qualitative property automatically and one has to use a shooting procedure. The iterative procedure is stopped to give the temperature and temperature gradient distributions when the boundary condition Eq.(6) is reached almost up the seventh decimal place. On the other hand, we would like to explore the influences of the step size Δx on the behaviour of the numerical solution because we are asked between the necessity of an accurate solution and the efficiency of a large step size over the interval [0,1]. Solutions with a low qualitative behaviour can be obtained as a consequence of the user specifying a Δx too lax; however, there is no defect of the numerical scheme for guaranteeing suitable results for $\theta(x)$ over [0,1]. We show in the next sections that our approach can deal with all these difficulties in a satisfactory way.

Certainly other schemes are possible, and perhaps to be preferred for specific kinds of problems, but ours has proved successful for all the numerical examples throughout the paper.

Case of *N*=1 and *m*=1

In this case, an analytical solution of the problem Eqs.(4)~(6) is available in the form:

$$\theta(x)=\frac{\mathrm{e}}{\mathrm{e}^2+1}\cdot(\mathrm{e}^x+\mathrm{e}^{-x}), \tag{9}$$

and the first derivative of Eq.(9) can be written as:

$$\theta'(x)=\frac{\mathrm{e}}{\mathrm{e}^2+1}\cdot(\mathrm{e}^x-\mathrm{e}^{-x}). \tag{10}$$

In Table 1 we show the comparisons between numerical solutions and the exact solutions for the case N=1 and m=1. It is found that the numerical results for two different step sizes Δx are very much close to the exact solution. The analysis shows that a finer grid does not affect much the accuracy of the numerical solution. This fact confirms the high accuracy of our approach, which provides us a simple way of treat this strongly non-linear problem. Realize that larger value of N in Eq.(4) corresponds to stronger non-linearity.

Table 1 Numerical results for $\theta(x)$ with two step size Δx when N=1 and m=1

x	$\theta(x)$		
	Δx=0.1	Δx=0.01	Exact
0.1	0.6512973	0.6512973	0.651297244
0.2	0.6610587	0.6610586	0.661058618
0.3	0.6774361	0.6774361	0.677436089
0.4	0.7005956	0.7005935	0.700593569
0.5	0.7307628	0.7307628	0.730762824
0.6	0.7682457	0.7682458	0.768245798
0.7	0.8134175	0.8134176	0.813417636
0.8	0.8667301	0.8667304	0.866730430
0.9	0.9287174	0.9287178	0.928717754
1.0	0.9999995	1.0000000	1.000000000

Some cases of practical interest

Although the solution for temperature has already been obtained by several investigators (e.g., Ganji *et al*., 2007; Tari *et al*., 2007), the numerical results for the heat transfer characteristics of engineering interest other than the fin tip temperature $\theta(0)$ have been reported very scarcely. New numerical results for the $\theta'(1)$ values and fin tip temperature $\theta(0)$ are presented here to compensate this gap. Realize that, as was indicated in (Chang, 2005), a typical physical quantity of interest is $Q_b=(\mathrm{d}\theta/\mathrm{d}x)_{x=1}$ (i.e., the first derivative of temperature θ at the fin root). This quantity is related with the amount of energy transferred from the fin base. It is found that our approach provides numerical results for Q_b, too.

In order to more fully characterize the behaviour of the numerical solutions with respect to both N and m parameters which govern this highly non-linear problem, representative dimensionless temperature and temperature gradient profiles at selected values of the exponent m when N=2 are shown in Fig.1; moreover, Fig.2 depicts those variations at selected values of the convective-conductive parameter N for m=4/3. It is shown that, as the fin parameter N increases, the fin tip temperature $\theta(0)$ decreases due to higher convective heat transfer to the surrounding fluid; however, the opposite trend is true for $\theta'(1)$. It is found from Tables 2 and 3 that the present approach gives fast and accurate numerical results for the two principal quantities of engineering interest, namely, $\theta(0)$ and $\theta'(1)$.

It is found from Table 3 that the parameter $Q_b=(\mathrm{d}\theta/\mathrm{d}x)_{x=1}$ increases with N for all the values m considered. However, for fixed N, Q_b decreases as m increases.

CONCLUSION

In this paper, an IVP approach is employed

Table 2 Numerical results when m=4/3 and N=2 with Δx=0.01

x	$\theta(x)$	$\theta'(x)$
0	0.51017858	0.0000000
0.1	0.5142625	0.0818217
0.2	0.5266013	0.1653970
0.3	0.5474608	0.2525452
0.4	0.5772944	0.3452199
0.5	0.6167608	0.4455842
0.6	0.6667491	0.5560956
0.7	0.7284126	0.6796042
0.8	0.8032146	0.8194736
0.9	0.8929859	0.9797261
1.0	1.0000000	1.1652260

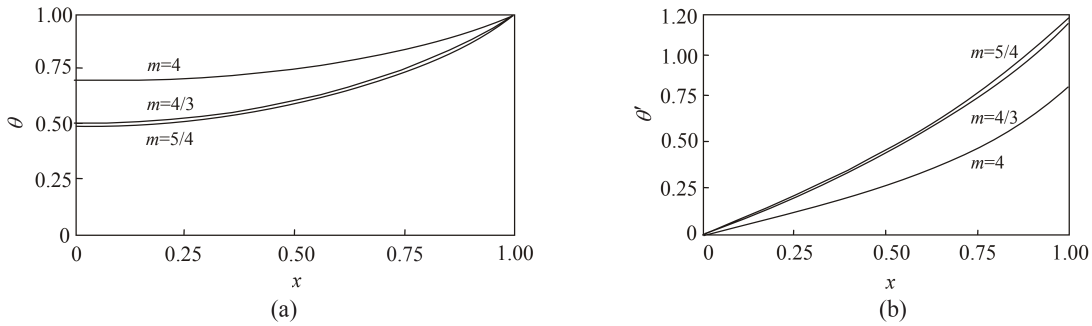

Fig.1 θ(a) and θ'(b) against x for several values of m when N=2

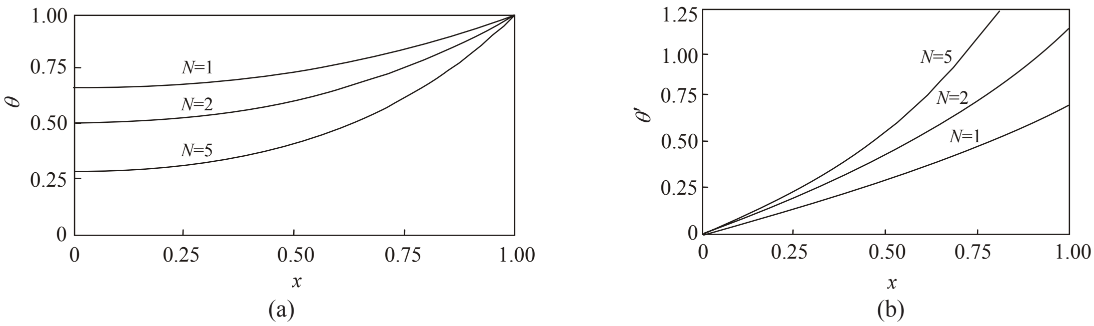

Fig.2 Plots of θ(a) and θ'(b) against x for several values of N when m=4/3

Table 3 Some $\theta(0)$ values for the problem under consideration. Parenthesis indicates the value of $Q_b=(\mathrm{d}\theta/\mathrm{d}x)_{x=1}$

m	N=1	N=2	N=5	N=10
5/4	0.66789780 (0.7283030)	0.49871670 (1.1858340)	0.27204914 (2.0510700)	0.14121898 (2.9631450)
4/3	0.67381427 (0.7183079)	0.51017858 (1.1652260)	0.28983128 (2.0118320)	0.15917473 (2.9075320)
3	0.75162203 (0.5834588)	0.64926864 (0.9068057)	0.50659704 (1.5281820)	0.40479059 (2.2058460)
4	0.77914525 (0.5339892)	0.69431843 (0.8190931)	0.57559625 (1.3688090)	0.48838578 (1.9720180)

to give numerical solutions to the complete and classical non-linear fin-type problem in heat transfer. The highly nonlinear ODE Eq.(4) satisfying the boundary conditions Eqs.(5) and (6) has been solved numerically using the 4th order Runge-Kutta shooting method for several values of the involved parameters, namely the fin parameter N and the exponent m. Approximate numerical solutions to strongly non-linear problems can be achieved without any assumption of linearization, and we can apply this approach to a variety of non-linear heat transfer problems. In this manner, a suitable helpfulness for engineer to analyze highly non-linear problems can be reached.

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