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Weighted approximation of functions with singularities by Bernstein operators

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Abstract: As an important type of polynomial approximation, approximation of functions by Bernstein operators is an important topic in approximation theory and computational theory. This paper gives global and pointwise estimates for weighted approximation of functions with singularities by Bernstein operators. The main results are the Jackson's estimates of functions $f\in(W_{w,\lambda})^2$ and $f\in C_w$, which extends the result of (Della Vecchia *et al*., 2004).

Key words: Bernstein operators, Functions with singularities, Weighted approximation, Pointwise estimates

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INTRODUCTION

A central topic in polynomial approximation is to connect the rate of approximation to smoothness properties of functions. The core of this theory lies in Jackson's theorem and its Stechkin-type converses. Recently, Mastroianni and Totik (1998; 1999; 2001) have generalized the classical Jackson's theorem to the weighted approximation of functions with singularities. It is well known that approximation of functions with singularities by polynomial is of special value in both theories and applications. As an important type of polynomial approximation, approximation of functions by Bernstein operators is an important topic in both approximation theory and computational theory, which plays an important role in neural networks, fitting date, curves, and surfaces. Some work has been done by (Della Vecchia *et al*., 2004; Yu and Zhao, 2006).

Denote by $C(I)$ the set of all continuous functions defined on the interval I. For any $f(x)\in C([0,1])$, the corresponding Bernstein operators are defined as follows:

$$B_n(f,x):=\sum_{k=0}^{n} f(k/n)p_{nk}(x), \tag{1}$$

where

$$p_{nk}(x):=\binom{n}{k}x^k(1-x)^{n-k},\ k=0,\ 1,\ \cdots,\ n.$$

Approximation of continuous functions by Bernstein operators has been extensively investigated (Ditzian and Totik, 1987).

Let

$$w(x)=x^{\alpha}(1-x)^{\beta},\ \alpha,\beta\ge 0,\ \alpha+\beta>0,\ 0\le x\le 1.$$

Set

$$C_w:=\left\{f(x)\in C\big((0,1)\big):\lim_{x\to 1}(wf)(x)=\lim_{x\to 0}(wf)(x)=0\right\}$$

equipped with the norm

$$\|wf\|_w:=\|wf\|:=\sup_{0\le x\le 1}\left|(wf)(x)\right|.$$

Define

$$W_{w,\lambda}^2:=\left\{f\in C_w:f'\in AC_{(0,1)},\ \|w\varphi^{2\lambda}f''\|<\infty\right\},$$

where $\varphi=\varphi(x)=\sqrt{x(1-x)}$.

To make the approximation of the functions $f\in C_w$ possible, we should modify Bernstein operators defined in Eq.(1) properly. Della Vecchia *et al*.(2004) introduced a kind of modified Bernstein operator as follows:

$$B_n^*(f,x):=(1-x)^n\left(2f\left(\frac{1}{n}\right)-f\left(\frac{1}{n}\right)\right)+\sum_{k=1}^{n-1}f\left(\frac{k}{n}\right)p_{nk}(x)$$
$$+x^n\left(2f\left(1-\frac{1}{n}\right)-f\left(1-\frac{2}{n}\right)\right),\qquad f\in C_w. \tag{2}$$

Define

$$\omega_{\varphi^\lambda}^2(f,t)_w:=\sup_{0<h\le t^*}\left\{\|\Delta_{h\varphi^\lambda}^2 f\|_{[t^*,1-t^*]}+\|w\overrightarrow{\Delta}_h^2 f\|_{[0,12t^*]}\right.$$
$$\left.+\|w\overleftarrow{\Delta}_h^2 f\|_{[1-12t^*,1]}\right\},$$

where $t^*=(2t)^{(1-\lambda/2)^{-1}}$, $0\le\lambda\le 1$, and

$$\overrightarrow{\Delta}_h^2 f(x):=f(x)-2f(x+h)+f(x+2h),$$
$$\overleftarrow{\Delta}_h^2 f(x):=f(x)-2f(x-h)+f(x-2h),$$
$$\Delta_{h\varphi^\lambda}^2 f(x):=f\left(x+\frac{h\varphi^\lambda(x)}{2}\right)-2f(x)+f\left(x-\frac{h\varphi^\lambda(x)}{2}\right).$$

Della Vecchia *et al*.(2004) established the following direct theorem:

Theorem 1 Let α, β>0, then

$$\|(w(f-B_n^*(f)))(x)\|\le\begin{cases}O\left(\|w\varphi^{2\lambda}f''\|/n\right), & f\in W_{w,1}^2,\\ O\left(\omega_\varphi^2(f,1/\sqrt{n})_w\right), & f\in C_w.\end{cases}$$

The main purpose of the present paper is to generalize Theorem 1 by establishing estimates combining the global estimates and pointwise estimates.

MAIN RESULT

We have the following result:

Theorem 2 Let α, β>0, $0\le\lambda\le1$, we have

(1) for any $f\in C_{w,\lambda}^2$,

$$\left|(w(f-B_n^*(f)))(x)\right|$$
$$=\begin{cases}O\left(\varphi^{2(1-\lambda)}(x)\|w\varphi^{2\lambda}f''\|/n\right), & \lambda+\alpha,\ \lambda+\beta>1,\\ O\left(\left(\delta_n^{1-\lambda}(x)/\sqrt{n}\right)^{4/(2-\lambda)}\|wf''\|\right.\\ \qquad\left.+\varphi^{2(1-\lambda)}(x)\|w\varphi^{2\lambda}f''\|/n\right), & \text{otherwise},\end{cases} \tag{3a}$$

where

$$\delta_n(x):=1/\sqrt{n}+\varphi(x),\quad n=1,\ 2,\ \cdots;$$

(2) for any $f\in C_w$,

$$\left|(w(f-B_n^*(f)))(x)\right|$$
$$\le\begin{cases}O\left(\omega_{\varphi^\lambda}^2\left(f,\varphi^{1-\lambda}(x)/\sqrt{n}\right)_w\right), & \lambda+\alpha,\lambda+\beta>1,\\ O\left(\omega_{\varphi^\lambda}^2\left(f,\delta_n^{1-\lambda}(x)/\sqrt{n}\right)_w\right), & \text{otherwise}.\end{cases} \tag{3b}$$

When λ=1, Theorem 2 becomes Theorem 1. When λ=0, we get the pointwise estimates, that is,

$$\left|(w(f-B_n^*(f)))(x)\right|$$
$$\le\begin{cases}O\left(\omega^2\left(f,\varphi(x)/\sqrt{n}\right)_w\right), & \lambda+\alpha,\lambda+\beta>1,\\ O\left(\omega^2\left(f,\delta_n(x)/\sqrt{n}\right)_w\right), & \text{otherwise}.\end{cases}$$

PROOF OF THEOREM 2

The proof of Theorem 2 is based on some preliminary lemmas. By symmetry, in what follows, we always assume $x\in[0, 1/2]$.

Lemma 1 (Yu and Zhou, in press) For any given γ>1, we have

(1) if $0\le x\le 1/\gamma$, $k\ge\gamma nx$, then

$$p_{nk}(x)\le(1/\gamma)^{k-\gamma nx}; \tag{4}$$

(2) if $0\le x\le 1/2$, $k\le nx/\gamma$, $nx\ge\gamma$, then

$$p_{nk}(x)\le(2/\gamma)^{nx/\gamma-k}. \tag{5}$$

Lemma 2 For any $\lambda\geq 0$, $x\in[0, 1/2]$, we have

$$E(x):=\sum_{k=0}^{n}p_{nk}(x)\frac{\varphi^{\lambda}(x)w(x)}{\varphi^{\lambda}(k^*/n)w(k^*/n)}=O(1),$$

where

$$k^*=\begin{cases}1, & k=0,\\ k, & 1<k<n,\\ n-1, & k=n.\end{cases}$$

Proof When $x=0$, the result is trivial. Now assume $0<x\leq 1/2$. First, we split $E(x)$ as follows:

$$E(x)=\sum_{|k/n-x|\leq x/2}+\sum_{|k/n-x|>x/2}=:E_1(x)+E_2(x). \tag{6}$$

If $|k/n-x|\leq x/2$ (thus $k\geq 1$), then

$$\varphi(x)\sim\varphi(k/n)\sim\varphi(k^*/n),$$
$$w(x)\sim w(k/n)\sim w(k^*/n),$$

hence

$$E_1(x)=O\left(\sum_{k=0}^{n}p_{nk}(x)\right)=O(1). \tag{7}$$

We further split $E_2(x)$ as follows [when $x>1/3$, set $E_{21}(x)=0$]:

$$E_2(x)=\sum_{3nx/2<k<n/2}+\sum_{n/2\leq k\leq 4n/5}+\sum_{4n/5<k\leq n-1}+\sum_{0\leq k\leq nx/5}+\sum_{nx/5<k<nx/2}$$
$$=:E_{21}(x)+E_{22}(x)+E_{23}(x)+E_{24}(x)+E_{25}(x). \tag{8}$$

Since

$$\varphi(k^*/n)\sim w(k^*/n)\sim 1,\quad \text{if } \frac{n}{2}\leq k\leq\frac{4n}{5},$$

$$\varphi(x)\sim\varphi(k^*/n),\ w(x)\sim w(k^*/n),\quad \text{if } \frac{nx}{5}<k<\frac{nx}{2},$$

then

$$E_{22}(x)=O\left(\sum_{k=0}^{n}p_{nk}(x)\right)=O(1), \tag{9}$$

and

$$E_{25}(x)=O\left(\sum_{k=0}^{n}p_{nk}(x)\right)=O(1). \tag{10}$$

When $3nx/2<k<n/2$, by noting that $k>3nx/2>4nx/3$, $(k-nx)/4\leq k-4nx/3$, and Eq.(4) with $\gamma=4/3$, we have

$$\begin{aligned}E_{21}(x)&=O(1)\sum\nolimits_{k>3nx/2}(3/4)^{k-4nx/3}(nx/k)^{\alpha+\lambda/2}\\&=O(1)\sum\nolimits_{k>3nx/2}(3/4)^{k-4nx/3}(1+|k-nx|)^{\alpha+\lambda/2}\\&=O(1)\sum\nolimits_{k>3nx/2}(3/4)^{k-4nx/3}(k-4nx/3)^{\alpha+\lambda/2}\\&=O(1).\end{aligned} \tag{11}$$

When $4n/5\leq k\leq n$, we have

$$k-4nx/3\geq 4n/5-4n/6=2n/15.$$

Therefore, by Eq.(4), we have

$$\begin{aligned}E_{23}(x)&=O(1)\sum\nolimits_{4n/5\leq k\leq n}(3/4)^{2n/15}\frac{x^{\alpha+\lambda/2}}{((n-k^*)/n)^{\beta+\lambda/2}}\\&=O(1)\sum\nolimits_{4n/5\leq k\leq n}(3/4)^{2n/15}n^{\beta+\lambda/2}\\&=O(1).\end{aligned} \tag{12}$$

We estimate $E_{24}(x)$ by considering the following two cases:

Case 1 $x\in[0, 5/n]$

In this case, it is obvious that

$$E_{24}(x)=O(p_{n0}(x)+p_{n1}(x))=O(1). \tag{13}$$

Case 2 $x\in(5/n, 1/2]$

By noting that $k\leq nx/5<nx/3$, $nx>3$, and Eq.(5) with $\gamma=3$, we have

$$\begin{aligned}E_{24}(x)&=O(1)\sum\nolimits_{0\leq k\leq nx/5}(2/3)^{nx/3-k}(nx/k^*)^{\alpha+\lambda/2}\\&=O(1)\sum\nolimits_{0\leq k\leq nx/5}(2/3)^{nx/3-k}(nx/(3k^*))^{\alpha+\lambda/2}\\&=O(1)\sum\nolimits_{0\leq k\leq nx/5}(2/3)^{nx/3-k}(1+|nx/3-k|)^{\alpha+\lambda/2}\\&=O(1).\end{aligned} \tag{14}$$

We complete the proof of Lemma 2 by combining Eqs.(6)~(14).

Set

$$\psi(x):=\begin{cases}0, & x\leq 0,\\ 10x^3-15x^4+6x^5, & 0<x<1,\\ 1, & x\geq 1.\end{cases}$$

Obviously, $\psi(x)$ is increasing and $\psi\in C^2_{(-\infty,\infty)}$. $\psi^{(i)}(0)=0$ for $i=0, 1, 2$; $\psi^{(i)}(1)=0$ for $i=1, 2$; $\psi(1)=1$. Let

$$P_1(f,x):=P_1(x)=(2-nx)f(1/n)+(nx-1)f(2/n),$$
$$P_2(f,x):=P_2(x)=(2-n(1-x))f(1-1/n)+(n(1-x)-1)f(1-2/n),$$
$$F_n(f,x):=F_n(x)=(1-\psi(nx-1))P_1(x)+(1-\psi(nx-n+2))\psi(nx-1)f(x)+\psi(nx-n+2)P_2(x)$$
$$=\begin{cases}P_1(x), & x\in[0,1/n],\\ (1-\psi(nx-1))P_1(x)+\psi(nx-1)f(x), & x\in[1/n,2/n],\\ f(x), & x\in[2/n,1-2/n],\\ (1-\psi(nx-n+2))\psi(nx-1)f(x)+\psi(nx-n+2)P_2(x), & x\in[1-2/n,1-1/n],\\ P_2(x), & x\in[1-1/n,1].\end{cases}$$

Evidently,

$$B_n^*(f,x)=B_n(F_n,x). \tag{15}$$

Lemma 3 Let $f\in W_{w,\lambda}^2$ and α, $\beta\geq 0$, then

$$\left|w(x)(F_n(f,x)-B_n(F_n,x))\right|=O\left(\varphi^{2(1-\lambda)}(x)\|w\varphi^{2\lambda}F_n''\|/n\right),\ x\in[0,\ 1]. \tag{16}$$

Proof When x=0, the result is trivial. Now assume $0<x\leq 1/2$. When $t\in[0,\ 1/n]\cup[1-1/n,\ 1]$, we have $F_n''(t)=0$. Thus

$$\left|\int_0^x (t-k/n)F_n''(t)\mathrm{d}t\right|=\left|\int_x^{1/n}(t-k/n)F_n''(t)\mathrm{d}t\right|,$$

and

$$\left|\int_x^1 (t-k/n)F_n''(t)\mathrm{d}t\right|=\left|\int_x^{1-1/n}(t-k/n)F_n''(t)\mathrm{d}t\right|.$$

By Lemma 1 and the following well-known inequality (Lorentz, 1953),

$$\sum_{k=0}^n p_{nk}(x)\left|x-k/n\right|^{\gamma}=O\left(n^{-\gamma/2}\varphi^{\gamma}(x)\right),$$

for any $x\in[1/n,\ 2/n]$, we have

$$\left|w(x)(F_n(f,x)-B_n(F_n,x))\right|$$
$$=\left|w(x)\sum_{k=0}^n p_{nk}(x)\int_x^{k/n}(t-k/n)F_n''(t)\mathrm{d}t\right|$$
$$\leq w(x)\sum_{k=0}^n p_{nk}(x)\left|\int_x^{k^*/n}(t-k/n)F_n''(t)\mathrm{d}t\right|$$
$$=O\left(\|w\varphi^{2\lambda}F_n''\|w(x)\sum_{k=0}^n\frac{p_{nk}(x)|x-k^*/n|^2}{\min\left\{(w\varphi^{2\lambda})(x),(w\varphi^{2\lambda})(k^*/n)\right\}}\right)$$
$$=O\left(\|w\varphi^{2\lambda}F_n''\|w(x)\left(\sum_{k=0}^n p_{nk}(x)|x-k/n|^4\right)^{1/2}\right.$$
$$\left.\cdot\left(\sum_{k=0}^n\frac{p_{nk}(x)}{\min\left\{(w^2\varphi^{4\lambda})(x),(w^2\varphi^{4\lambda})(k^*/n)\right\}}\right)^{1/2}\right)$$
$$=O\left(w(x)\varphi^2(x)\|w\varphi^{2\lambda}F_n''\|/n\right)$$
$$\cdot\left(\sum_{k=0}^n\frac{p_{nk}(x)}{\min\left\{(w^2\varphi^{4\lambda})(x),(w^2\varphi^{4\lambda})(k^*/n)\right\}}\right)^{1/2}$$
$$=O\left(\frac{\varphi^{2(1-\lambda)}(x)}{n}\|w\varphi^{2\lambda}F_n''\|\left(1+\sum_{k=0}^n\frac{p_{nk}(x)(w^2\varphi^{4\lambda})(x)}{(w^2\varphi^{4\lambda})(k^*/n)}\right)^{1/2}\right)$$
$$=O(\varphi^{2(1-\lambda)}(x)\|w\varphi^{2\lambda}F_n''\|/n).$$

On the other hand, when $x\in[0,\ 1/n]$, noting that

$$\int_0^x (t-k/n)F_n''(t)\mathrm{d}t=0,$$

we can obtain the result by using $\sum_{k=1}^n$ to replace $\sum_{k=0}^n$ in the above deduction.

Lemma 4 Let $f\in W_{w,\lambda}^2$, $0\leq\lambda\leq 1$, then for $x\in[0,\ 2/n]$, we have

$$\left|w(x)(f(x)-P_1(x))\right|=\begin{cases}O\left(\varphi^{2(1-\lambda)}(x)\|w\varphi^{2\lambda}f''\|_{[0,2/n]}/n\right), & \lambda+\alpha>1,\\ O\left(\left(\delta_n^{1-\lambda}(x)/\sqrt{n}\right)^{4/(2-\lambda)}\|wf''\|_{[0,2/n]}\right), & \text{otherwise}.\end{cases} \tag{17}$$

For $x\in[1-2/n,\ 1]$, we have

$$\left|w(x)(f(x)-P_2(x))\right|=\begin{cases}O\left(\varphi^{2(1-\lambda)}(x)\|w\varphi^{2\lambda}f''\|_{[1-2/n,1]}/n\right), & \lambda+\beta>1,\\ O\left(\left(\delta_n^{1-\lambda}(x)/\sqrt{n}\right)^{4/(2-\lambda)}\|wf''\|_{[1-2/n,1]}\right), & \text{otherwise}.\end{cases} \tag{18}$$

Proof By symmetry, we only need to prove Eq.(17). If $f\in W_{w,\lambda}^2$, then for any $x\in[0,\ 2/n]$,

$$f(2/n)=f(x)+f'(x)(2/n-x)+\int_{2/n}^{x}(t-2/n)f''(t)\mathrm{d}t,$$

$$f(1/n)=f(x)+f'(x)(1/n-x)+\int_{1/n}^{x}(t-1/n)f''(t)\mathrm{d}t,$$

$$\delta_n(x)\sim 1/\sqrt{n},\quad n=1,\ 2,\ \cdots,$$

thus

$$\begin{aligned}&|w(x)(f(x)-P_1(x))|\\ &\le\left|w(x)(nx-1)\int_{2/n}^{x}(t-2/n)f''(t)\mathrm{d}t\right|\\ &\quad+\left|w(x)(2-nx)\int_{1/n}^{x}(t-1/n)f''(t)\mathrm{d}t\right|\\ &=O\left(\|wf''\|_{[0,2/n]}/n^2\right)\\ &=O\left(\left(\delta_n^{1-\lambda}(x)/\sqrt{n}\right)^{4/(2-\lambda)}\|wf''\|_{[0,2/n]}\right),\end{aligned}$$

which implies the second inequality of Eq.(17).

When $\lambda+\alpha>1$, we have

$$\begin{aligned}&\left|w(x)(2-nx)\int_{1/n}^{x}(t-1/n)f''(t)\mathrm{d}t\right|\\ &=O\left(\|w\varphi^{2\lambda}f''\|_{[0,2/n]}\frac{w(x)}{n}\int_{x}^{1/n}t^{-\lambda-\alpha}\mathrm{d}t\right)\\ &=O\left(\varphi^{2(1-\lambda)}(x)\|w\varphi^{2\lambda}f''\|_{[0,2/n]}/n\right).\end{aligned}$$

Similarly, we have

$$\begin{aligned}&\left|w(x)(nx-1)\int_{2/n}^{x}(t-2/n)f''(t)\mathrm{d}t\right|\\ &=O\left(\varphi^{2(1-\lambda)}(x)\|w\varphi^{2\lambda}f''\|_{[0,2/n]}/n\right).\end{aligned}$$

Therefore, the first inequality of Eq.(17) also holds.

Lemma 5 If $f\in W_{w,\lambda}^2$, $0\le\lambda\le1$, then

$$\|w\varphi^{2\lambda}F_n''\|=O(\|w\varphi^{2\lambda}f''\|).\tag{19}$$

Proof Since $F_n''(x)=0$ for any $0\le x<1/n$, and $F_n\equiv f$ for any $2/n\le x<1/2$, we only need to consider the case when $x\in[1/n, 2/n]$. In this case, $\varphi(x)\sim1/\sqrt{n}$, hence observing the proof of the first inequality of Eq.(17), for any $0\le\lambda\le1$, $\alpha\ge0$, we have

$$\begin{aligned}&|w(x)(f(x)-P_1(x))|\\ &=O\left((\varphi^{2(1-\lambda)}(x))\|w\varphi^{2\lambda}f''\|_{[1/n,2/n]}/n\right).\end{aligned}\tag{20}$$

Since

$$F_n(x)=P_1(x)+\psi(nx-1)(f(x)-P_1(x)),$$

then

$$\begin{aligned}F_n''(x)&=n^2\psi''(nx-1)(f(x)-P_1(x))+2n\psi'(nx-1)\\ &\quad\cdot(f(x)-P_1(x))'+\psi(nx-1)f''(x)\\ &=:I_1(x)+I_2(x)+I_3(x).\end{aligned}$$

By Eq.(20) and the definition of $\psi(x)$,

$$\begin{aligned}&|(w\varphi^{2\lambda})(x)I_1(x)|\\ &=O\left(n^2(w\varphi^{2\lambda})(x)|f(x)-P_1(x)|\right)\\ &=O\left(n^2\varphi^2(x)\|w\varphi^{2\lambda}f''\|_{[0,2/n]}\right)\\ &=O\left(n^2(w\varphi^{2\lambda})(x)|f(x)-P_1(x)|\right)\\ &=O(\|w\varphi^{2\lambda}f''\|).\end{aligned}$$

For $I_3(x)$, it is obvious that

$$|(w\varphi^{2\lambda})(x)I_3(x)|=O(\|w\varphi^{2\lambda}f''\|).$$

Finally,

$$\begin{aligned}&|(w\varphi^{2\lambda})(x)I_2(x)|\\ &=O\left(n(w\varphi^{2\lambda})(x)|f'(x)-P_1'(x)|\right)\\ &=O\left(n(w\varphi^{2\lambda})(x)\left|f'(x)-n\int_{1/n}^{2/n}f'(t)\mathrm{d}t\right|\right)\\ &=O\left(n(w\varphi^{2\lambda})(x)\left|n\int_{1/n}^{2/n}\int_{t}^{x}f''(u)\mathrm{d}u\mathrm{d}t\right|\right)\\ &=O\left(n(w\varphi^{2\lambda})(x)\left|\int_{1/n}^{2/n}f''(u)\mathrm{d}u\right|\right)\\ &=O(\|w\varphi^{2\lambda}f''\|).\end{aligned}$$

Lemma 6 (Theorem 1 in (Della Vecchia *et al*., 2004)) For any α, $\beta>0$, $f\in C_w$,

$$\|wB_n^*(f)\|=O(\|wf\|).$$

Proof (of Theorem 2) By Lemmas 3~5, we have

$$\begin{aligned}&\|w(f-B_n^*(f,x))\|\\ &\le\|w(f-F_n(f))\|+\|w(F_n(f)-B_n(F_n(f)))\|\\ &\le\|w(f-F_n(f))\|+O\left(\varphi^{2(1-\lambda)}(x)\|w\varphi^{2\lambda}f''\|/n\right)\\ &=\|w(f-F_n(f))\|_{[0,2/n]}+\|w(f-F_n(f))\|_{[1-2/n,1]}\\ &\quad+O\left(\varphi^{2(1-\lambda)}(x)\|w\varphi^{2\lambda}f''\|/n\right)\end{aligned}$$

$$=\begin{cases} O\left(\varphi^{2(1-\lambda)}(x)\,\|w\varphi^{2\lambda}f''\|/n\right), & \lambda+\alpha, \lambda+\beta>1, \\ O\left(\left(\delta_n^{1-\lambda}(x)/\sqrt{n}\right)^{4/(2-\lambda)}\|wf''\| \right. \\ \qquad \left. +\varphi^{2(1-\lambda)}(x)\,\|w\varphi^{2\lambda}f''\|/n\right), & \text{otherwise,} \end{cases}$$

which implies Eq.(3a).

Define

$$K^2_{\varphi^\lambda}(f,t^2)_w := \inf_{g'\in AC_{\mathrm{loc}}}\left\{\|f-g\|_w + t^2\|\varphi^{2\lambda}g''\|_w\right\}.$$

By Theorem 6.1.1 in (Ditzian and Totik, 1987), we have

$$K^2_{\varphi^\lambda}(f,t^2)_w \sim \omega^2_{\varphi^\lambda}(f,t)_w.$$

Taking $t=\varphi^{1-\lambda}(x)/\sqrt{n}$ and $g\in W^2_{w,\lambda}$, by Lemma 6 and the first inequality of Eq.(3a), we see that for any $\lambda+\alpha$, $\lambda+\beta>1$,

$$\begin{aligned}&\|w(f-B_n^*(f,x))\|\\ &\le \|w(f-g)\| + \|w(B_n^*(f-g))\|\cdot\|w(g-B_n^*(g))\|\\ &= O\left(\|w(f-g)\| + \varphi^{2(1-\lambda)}(x)\,\|w\varphi^{2\lambda}g''\|/n\right)\\ &= O\left(\omega^2_{\varphi^\lambda}(f,\varphi^{1-\lambda}(x)/\sqrt{n})_w\right).\end{aligned}$$

Define

$$\overline{K}^2_{\varphi^\lambda}(f,t^2)_w := \inf_{g'\in AC_{\mathrm{loc}}}\left\{\|f-g\|_w + t^2\|\varphi^{2\lambda}g''\|_w + t^{4/(2-\lambda)}\|g''\|_w\right\}.$$

By Guo *et al.*(2003), we have

$$\overline{K}^2_{\varphi^\lambda}(f,t^2)_w \sim K^2_{\varphi^\lambda}(f,t^2)_w.$$

Therefore

$$\overline{K}^2_{\varphi^\lambda}(f,t^2)_w \sim K^2_{\varphi^\lambda}(f,t^2)_w \sim \omega^2_{\varphi^\lambda}(f,t)_w.$$

Taking $t=\delta_n^{1-\lambda}(x)/\sqrt{n}$, by the second inequality of Eq.(3a), repeating the above deduction, we see that for any $0\le\lambda\le1$, α, $\beta>0$,

$$\|w(f-B_n^*(f,x))\| = O\left(\omega^2_{\varphi^\lambda}(f,\delta_n^{1-\lambda}(x)/\sqrt{n})_w\right).$$

Therefore, we finish the proof of Eq.(3b).

CONCLUSION

This paper gives the Jackson's theorem to the weighted approximation of functions with singularities by one kind of modified Bernstein operators. The main result (i.e., Theorem 2) of this paper generalizes the result of (Della Vecchia *et al.*, 2004) by establishing estimates combining the global and pointwise estimates. The techniques used in this paper can be applied to more such problems.

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