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Optimal operating policy for a controllable queueing model with a fuzzy environment

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Abstract: We construct the membership functions of the fuzzy objective values of a controllable queueing model, in which cost elements, arrival rate and service rate are all fuzzy numbers. Based on Zadeh's extension principle, a set of parametric nonlinear programs is developed to find the upper and lower bounds of the minimal average total cost per unit time at the possibility level. The membership functions of the minimal average total cost are further constructed using different values of the possibility level. A numerical example is solved successfully to illustrate the validity of the proposed approach. Because the object value is expressed and governed by the membership functions, the optimization problem in a fuzzy environment for the controllable queueing models is represented more accurately and analytical results are more useful for system designers and practitioners.

Key words: Controllable queue, Fuzzy sets, Membership function, Nonlinear programming
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INTRODUCTION

Queueing models with a control operating policy are effective approaches for performance analysis of computer and telecommunication systems, manufacturing/production systems and inventory control (Kleinrock, 1975; Kella, 1989; Lee and Srinivasan, 1989; Buzacott and Shanthikumar, 1993; Gross and Harris, 1998). One of the well-known control operating policies of queueing models is the N policy, meaning that the server is turned on when N ($N\geq1$) or more units are in presence, and off when the system is empty (Yadin and Naor, 1963). Optimality of N policy (the optimal value of N) has also been shown by many researchers (Kella, 1989; Lee *et al*., 1994a; 1994b; 1995; Lee and Park, 1997; Pearn *et al*., 2004; Wang *et al*., 2004; Arumuganathan and Jeyakumar, 2005; Choudhury and Madan, 2005). This has also given rise to a vast and rich literature, from which we can review the survey by Tadj and Choudhury (2005). Recently Tadj *et al*.(2006a; 2006b) used an embedded Markov chain to investigate an $M/G/1$ quorum queueing system with N policy and Bernoulli vacation schedule. They also performed optimization work on thresholds of the quorum N policy. Choudhury and Paul (2006) investigated the N policy $M^{[x]}/G/1$ queueing system with a second optional service channel and further derived a procedure to obtain an optimal operating policy using a linear cost structure. Note that the cost function of control operating policies for queueing models mainly depends on the holding cost of a customer in the system and the setup cost of a server (Yadin and Naor, 1963; Heyman, 1968; Bell, 1971). The optimal value of N attempts to minimize the sum of the two main costs.

Existing research works, including those mentioned above, have been developed to search for the optimal operating policy of queueing models when the cost elements, and the arrival and service patterns are known exactly. However, in many real-world applications, these parameters may not be estimated precisely. For example, the holding cost of a customer and the setup cost of a server are more suitably described in linguistic terms such as high, moderate, or low. In fact the main barrier to implementing cost models for controllable queueing systems is that it may be difficult to obtain a reliable estimate of the holding cost of a customer and the setup cost of a server (Hillier and Lieberman, 2001; Taha, 2003). Moreover, the arrival and service patterns may only be characterized subjectively; that is, the arrival and service are typically described in everyday language summaries with a central tendency, such as "the mean arrival rate is around 5 per day", or "the mean service rate is about 10 per hour", rather than with complete probability distributions. In other words, these system parameters are both possibilistic and probabilistic. Thus, the control operating policy of queueing models with fuzzy parameters is potentially much more useful and realistic than the commonly used crisp queues (Zadeh, 1978; Li and Lee, 1989). By extending the usual crisp queues with a control operating policy to fuzzy environments, such queueing models become appropriate for a wider range of applications (Zhang *et al*., 2005). In contrast, Zhang (2006) studied the batch scheduling problem using fuzzy-logic-based methodology. Buckley *et al*.(2001) treated the queueing design problems of an *M*/*M*/*c* queueing system with fuzzy arrival rate and service rate, in which fuzzy cost elements are superimposed on the descriptive models to optimize *c*.

The queueing control problems with fuzzy environment are considered in this paper. We develop an approach that provides the objective value for the controllable queue with a control operating policy and fuzzy parameters, i.e., fuzzified exponential arrival rate, service rate, and fuzzy cost elements. Through α-cuts and Zadeh's extension principle, we transform the fuzzy objective value to a family of crisp objective values. As α varies, the family of crisp objective values is described and solved using parametric nonlinear programming (NLP). The NLP solutions completely and successfully yield the membership functions of the average cost per unit time.

MODEL DESCRIPTION

Consider a queueing system in which arrivals occur at a single-server facility as a Poisson process with rate λ, and all service time is independent and identically distributed according to an exponential distribution with rate μ. After all the customers are served in the queue exhaustively, the server operates *N* policy. The server reactivates as soon as the number of customers in the queue reaches a threshold *N* since the idle period initiates. In practical use, we often search for the optimum value of the control threshold *N*, say N^*, to minimize the following function:

$$F(N) = C_{\mathrm{h}} L(\lambda, \mu, N) + C_{\mathrm{s}} / \Omega(\lambda, \mu, N), \quad (1)$$

where $\Omega(\lambda, \mu, N) = N\mu / [\lambda(\mu-\lambda)]$ is the average length of a busy cycle, $L(\lambda, \mu, N)=(N-1)/2+\lambda/\mu+\mu/(\mu-\lambda)$ is the average system size, C_{h} is the holding cost per unit time for each customer (unit) in the system, and C_{s} is the setup cost per busy cycle (Yadin and Naor, 1963; Heyman, 1968; Choudhury and Paul, 2006).

The cost minimization problem in Eq.(1) can be illustrated mathematically as

$$\Psi(\lambda, \mu, C_{\mathrm{h}}, C_{\mathrm{s}}) = \min_{N>0} \{ C_{\mathrm{h}} L(\lambda, \mu, N) + C_{\mathrm{s}} / \Omega(\lambda, \mu, N) \}. \quad (2)$$

Suppose that the arrival rate λ, service rate μ, holding cost C_{h}, and setup cost C_{s} are approximately known and can be represented by the fuzzy sets $\tilde{\lambda}, \tilde{\mu}, \tilde{C}_{\mathrm{h}}$ and $\tilde{C}_{\mathrm{s}}$, respectively. Let $\eta_{\tilde{\lambda}}(x)$, $\eta_{\tilde{\mu}}(y)$, $\eta_{\tilde{C}_{\mathrm{h}}}(w)$ and $\eta_{\tilde{C}_{\mathrm{s}}}(v)$ denote the membership functions of $\tilde{\lambda}$, $\tilde{\mu}$, $\tilde{C}_{\mathrm{h}}$ and $\tilde{C}_{\mathrm{s}}$, respectively. We then have the following fuzzy sets:

$$\tilde{\lambda} = \{(x, \eta_{\tilde{\lambda}}(x)) \mid x \in X\}, \quad (3a)$$

$$\tilde{\mu} = \{(y, \eta_{\tilde{\mu}}(y)) \mid y \in Y\}, \quad (3b)$$

$$\tilde{C}_{\mathrm{h}} = \{(w, \eta_{\tilde{C}_{\mathrm{h}}}(w)) \mid w \in W\}, \quad (3c)$$

$$\tilde{C}_{\mathrm{s}} = \{(v, \eta_{\tilde{C}_{\mathrm{s}}}(v)) \mid v \in V\}, \quad (3d)$$

where *X*, *Y*, *W* and *V* are the crisp universal sets of the arrival rate, service rate, holding cost and setup cost, respectively. Note that a fuzzy set $\tilde{\Theta}$ in its universal set *S* is convex if $\eta_{\tilde{\Theta}}(\delta s_1 + (1-\delta)s_2) \geq \min\{\eta_{\tilde{\Theta}}(s_1), \eta_{\tilde{\Theta}}(s_2)\}$, where $\eta_{\tilde{\Theta}}$ is its membership function, $s_1, s_2 \in S$ and $\delta \in [0, 1]$ (Zimmermann, 2001).

Following Zadeh's extension principle (Zadeh, 1978; Li and Lee, 1989), the membership function of the objective value $\tilde{\Psi}(\tilde{\lambda}, \tilde{\mu}, \tilde{C}_{\mathrm{h}}, \tilde{C}_{\mathrm{s}})$ from Eq.(2) is defined as

$$\eta_{\tilde{\Psi}(\tilde{\lambda},\tilde{\mu},\tilde{C}_h,\tilde{C}_s)}(z)= \sup_{\substack{x\in X,y\in Y\\ w\in W,v\in V}} \min_{z=\Psi(x,y,w,v)} \left\{\eta_{\tilde{\lambda}}(x),\eta_{\tilde{\mu}}(y),\eta_{\tilde{C}_h}(w),\eta_{\tilde{C}_s}(v)\right\}, \quad (4)$$

where

$$\Psi(x,y,w,v)= \min_{N>0}\left\{w\frac{(N-1)y(y-x)+2(y-x)+2y^2}{2y(y-x)}+v\frac{x(y-x)}{Ny}\right\}.$$

Clearly, the minimal average total cost per unit time $\tilde{\Psi}$ is not a crisp number but a fuzzy number. However, it is not easy to obtain its value. We are interested in deriving its membership function $\eta_{\tilde{\Psi}}(z)$. In this study we approach the representation problem using a mathematical programming technique. Parametric NLP is developed to find the α-cuts of $\tilde{\Psi}$ based on the extension principle.

PARAMETRIC NONLINEAR PROGRAMMING APPROACH

One approach is to construct the membership function on $\eta_{\tilde{\Psi}}$ by deriving the α-cuts (or α-level sets) of $\tilde{\Psi}$. Denote the α-level sets of $\tilde{\lambda}, \tilde{\mu}, \tilde{C}_h$ and $\tilde{C}_s$ as

$$\lambda(\alpha)=[x_\alpha^L,\ x_\alpha^U] =[\min_{x\in X}\{x\,|\,\eta_{\tilde{\lambda}}(x)\geq\alpha\},\ \max_{x\in X}\{x\,|\,\eta_{\tilde{\lambda}}(x)\geq\alpha\}], \quad (5a)$$

$$\mu(\alpha)=[y_\alpha^L,\ y_\alpha^U] =[\min_{y\in Y}\{y\,|\,\eta_{\tilde{\mu}}(y)\geq\alpha\},\max_{y\in Y}\{y\,|\,\eta_{\tilde{\mu}}(y)\geq\alpha\}], \quad (5b)$$

$$C_h(\alpha)=[w_\alpha^L,\ w_\alpha^U] =[\min_{w\in W}\{w\,|\,\eta_{\tilde{C}_h}(w)\geq\alpha\},\ \max_{w\in W}\{w\,|\,\eta_{\tilde{C}_h}(w)\geq\alpha\}], \quad (5c)$$

$$C_s(\alpha)=[v_\alpha^L,\ v_\alpha^U] =[\min_{v\in V}\{v\,|\,\eta_{\tilde{C}_s}(v)\geq\alpha\},\ \max_{v\in V}\{v\,|\,\eta_{\tilde{C}_s}(v)\geq\alpha\}]. \quad (5d)$$

The arrival rate, service rate, holding cost and setup cost are shown as intervals when the membership functions are no less than a given possibility level of α. As a result, the bounds of these intervals can be described as functions of α by $x_\alpha^L=\min\eta_{\tilde{\lambda}}^{-1}(\alpha)$, $x_\alpha^U=\max\eta_{\tilde{\lambda}}^{-1}(\alpha)$, $y_\alpha^L=\min\eta_{\tilde{\mu}}^{-1}(\alpha)$, $y_\alpha^U=\max\eta_{\tilde{\mu}}^{-1}(\alpha)$, $w_\alpha^L=\min\eta_{\tilde{C}_h}^{-1}(\alpha)$, $w_\alpha^U=\max\eta_{\tilde{C}_h}^{-1}(\alpha)$, $v_\alpha^L=\min\eta_{\tilde{C}_s}^{-1}(\alpha)$ and $v_\alpha^U=\max\eta_{\tilde{C}_s}^{-1}(\alpha)$, respectively. Therefore, we can use the α-cuts of $\tilde{\Psi}$ to construct its membership function since the membership function defined in Eq.(4) is parameterized by α.

Using Zadeh's extension principle, $\eta_{\tilde{\Psi}}(z)$ is the minimum of $\eta_{\tilde{\lambda}}(x)$, $\eta_{\tilde{\mu}}(y)$, $\eta_{\tilde{C}_h}(w)$ and $\eta_{\tilde{C}_s}(v)$. To derive the membership function $\eta_{\tilde{\Psi}}(z)$, we need at least one of the following cases to hold such that $z=\Psi(x,y,w,v)$ satisfies $\eta_{\tilde{\Psi}}(z)=\alpha$:

Case (i): $\eta_{\tilde{\lambda}}(x)=\alpha,\eta_{\tilde{\mu}}(y)\geq\alpha,\eta_{\tilde{C}_h}(w)\geq\alpha,\eta_{\tilde{C}_s}(v)\geq\alpha$,

Case (ii): $\eta_{\tilde{\lambda}}(x)\geq\alpha,\eta_{\tilde{\mu}}(y)=\alpha,\eta_{\tilde{C}_h}(w)\geq\alpha,\eta_{\tilde{C}_s}(v)\geq\alpha$,

Case (iii): $\eta_{\tilde{\lambda}}(x)\geq\alpha,\eta_{\tilde{\mu}}(y)\geq\alpha,\eta_{\tilde{C}_h}(w)=\alpha,\eta_{\tilde{C}_s}(v)\geq\alpha$,

Case (iv): $\eta_{\tilde{\lambda}}(x)\geq\alpha,\eta_{\tilde{\mu}}(y)\geq\alpha,\eta_{\tilde{C}_h}(w)\geq\alpha,\eta_{\tilde{C}_s}(v)=\alpha$.

This can be accomplished by the parametric NLP techniques. Let

$$F(N;x,y,w,v) = w\frac{(N-1)y(y-x)+2(y-x)+2y^2}{2y(y-x)}+v\frac{x(y-x)}{Ny}.$$

For Case (i), the corresponding parametric nonlinear programs for finding the lower and upper bounds of the α-cut of $\eta_{\tilde{\Psi}}(z)$ are

$$(\Psi_\alpha^L)_1=\min_{x,y,w,v\in\mathbb{R}^+}\min_{N>0}F(N;x,y,w,v),$$
$$\text{s.t. } x_\alpha^L\leq x\leq x_\alpha^U,\ y\in\mu(\alpha),\ w\in C_h(\alpha),\ v\in C_s(\alpha); \quad (6a)$$

$$(\Psi_\alpha^U)_1=\max_{x,y,w,v\in\mathbb{R}^+}\min_{N>0}F(N;x,y,w,v),$$
$$\text{s.t. } x_\alpha^L\leq x\leq x_\alpha^U,\ y\in\mu(\alpha),\ w\in C_h(\alpha),\ v\in C_s(\alpha). \quad (6b)$$

For Case (ii), are

$$(\Psi_\alpha^L)_2=\min_{x,y,w,v\in\mathbb{R}^+}\min_{N>0}F(N;x,y,w,v),$$
$$\text{s.t. } x\in\lambda(\alpha),\ y_\alpha^L\leq y\leq y_\alpha^U,\ w\in C_h(\alpha),\ v\in C_s(\alpha); \quad (7a)$$

$$(\Psi_\alpha^U)_2=\max_{x,y,w,v\in\mathbb{R}^+}\min_{N>0}F(N;x,y,w,v),$$
$$\text{s.t. } x\in\lambda(\alpha),\ y_\alpha^L\leq y\leq y_\alpha^U,\ w\in C_h(\alpha),\ v\in C_s(\alpha). \quad (7b)$$

For Case (iii), are

$$(\Psi_\alpha^{\mathrm{L}})_3 = \min_{x,y,w,v\in\mathbb{R}^+} \min_{N>0} F(N;x,y,w,v), \quad \text{s.t. } x\in\lambda(\alpha),\ y\in\mu(\alpha),\ w_\alpha^{\mathrm{L}}\le w\le w_\alpha^{\mathrm{U}},\ v\in C_{\mathrm{s}}(\alpha); \tag{8a}$$

$$(\Psi_\alpha^{\mathrm{U}})_3 = \max_{x,y,w,v\in\mathbb{R}^+} \min_{N>0} F(N;x,y,w,v), \quad \text{s.t. } x\in\lambda(\alpha),\ y\in\mu(\alpha),\ w_\alpha^{\mathrm{L}}\le w\le w_\alpha^{\mathrm{U}},\ v\in C_{\mathrm{s}}(\alpha). \tag{8b}$$

And for Case (iv), are

$$(\Psi_\alpha^{\mathrm{L}})_4 = \min_{x,y,w,v\in\mathbb{R}^+} \min_{N>0} F(N;x,y,w,v), \quad \text{s.t. } x\in\lambda(\alpha),\ y\in\mu(\alpha),\ w\in C_{\mathrm{h}}(\alpha),\ v_\alpha^{\mathrm{L}}\le v\le v_\alpha^{\mathrm{U}}; \tag{9a}$$

$$(\Psi_\alpha^{\mathrm{U}})_4 = \max_{x,y,w,v\in\mathbb{R}^+} \min_{N>0} F(N;x,y,w,v), \quad \text{s.t. } x\in\lambda(\alpha),\ y\in\mu(\alpha),\ w\in C_{\mathrm{h}}(\alpha),\ v_\alpha^{\mathrm{L}}\le v\le v_\alpha^{\mathrm{U}}. \tag{9b}$$

From the definition of $\lambda(\alpha)$, $\mu(\alpha)$, $C_{\mathrm{h}}(\alpha)$ and $C_{\mathrm{s}}(\alpha)$ in Eq.(5), $x\in\lambda(\alpha)$, $y\in\mu(\alpha)$, $w\in C_{\mathrm{h}}(\alpha)$ and $v\in C_{\mathrm{s}}(\alpha)$ can be replaced by $x\in[x_\alpha^{\mathrm{L}}, x_\alpha^{\mathrm{U}}]$, $y\in[y_\alpha^{\mathrm{L}}, y_\alpha^{\mathrm{U}}]$, $w\in[w_\alpha^{\mathrm{L}}, w_\alpha^{\mathrm{U}}]$ and $v\in[v_\alpha^{\mathrm{L}}, v_\alpha^{\mathrm{U}}]$, respectively. All α-cuts form a nested structure with respect to α (Kaufmann, 1975; Zimmermann, 2001); i.e., given $0<\alpha_2<\alpha_1\le 1$, we have $[x_{\alpha_1}^{\mathrm{L}}, x_{\alpha_1}^{\mathrm{U}}]\subseteq[x_{\alpha_2}^{\mathrm{L}}, x_{\alpha_2}^{\mathrm{U}}]$, $[y_{\alpha_1}^{\mathrm{L}}, y_{\alpha_1}^{\mathrm{U}}]\subseteq[y_{\alpha_2}^{\mathrm{L}}, y_{\alpha_2}^{\mathrm{U}}]$, $[w_{\alpha_1}^{\mathrm{L}}, w_{\alpha_1}^{\mathrm{U}}]\subseteq[w_{\alpha_2}^{\mathrm{L}}, w_{\alpha_2}^{\mathrm{U}}]$ and $[v_{\alpha_1}^{\mathrm{L}}, v_{\alpha_1}^{\mathrm{U}}]\subseteq[v_{\alpha_2}^{\mathrm{L}}, v_{\alpha_2}^{\mathrm{U}}]$. Therefore, Eqs.(6a), (7a), (8a) and (9a) have the same smallest element and Eqs.(6b), (7b), (8b) and (9b) have the same largest element. To find the membership function $\eta_{\tilde{\Psi}}(z)$, it suffices to find the left shape function and the right shape function of $\eta_{\tilde{\Psi}}(z)$, which is equivalent to finding the lower bound Ψ_α^{L} and upper bound Ψ_α^{U} of the α-cuts of $\eta_{\tilde{\Psi}}$. Based on Eq.(4), Ψ_α^{L} and Ψ_α^{U} can be rewritten as

$$\Psi_\alpha^{\mathrm{L}} = \min_{x,y,w,v\in\mathbb{R}^+} \min_{N>0} F(N;x,y,w,v), \quad \text{s.t. } x_\alpha^{\mathrm{L}}\le x\le x_\alpha^{\mathrm{U}},\ y_\alpha^{\mathrm{L}}\le y\le y_\alpha^{\mathrm{U}},\ w_\alpha^{\mathrm{L}}\le w\le w_\alpha^{\mathrm{U}},\ v_\alpha^{\mathrm{L}}\le v\le v_\alpha^{\mathrm{U}}; \tag{10a}$$

$$\Psi_\alpha^{\mathrm{U}} = \max_{x,y,w,v\in\mathbb{R}^+} \min_{N>0} F(N;x,y,w,v), \quad \text{s.t. } x_\alpha^{\mathrm{L}}\le x\le x_\alpha^{\mathrm{U}},\ y_\alpha^{\mathrm{L}}\le y\le y_\alpha^{\mathrm{U}},\ w_\alpha^{\mathrm{L}}\le w\le w_\alpha^{\mathrm{U}},\ v_\alpha^{\mathrm{L}}\le v\le v_\alpha^{\mathrm{U}}. \tag{10b}$$

Since Eq.(10a) is to find the minimum of all the minimum objective values, we can simplify the two-level mathematical program to the following traditional one-level mathematical program:

$$\Psi_\alpha^{\mathrm{L}} = \min_{N>0} F(N;x,y,w,z), \quad \text{s.t. } x_\alpha^{\mathrm{L}}\le x\le x_\alpha^{\mathrm{U}},\ y_\alpha^{\mathrm{L}}\le y\le y_\alpha^{\mathrm{U}},\ w_\alpha^{\mathrm{L}}\le w\le w_\alpha^{\mathrm{U}},\ v_\alpha^{\mathrm{L}}\le v\le v_\alpha^{\mathrm{U}}. \tag{11}$$

In order to solve Eq.(10b), we treated the inner $\min_N$ problem, in that x, y, w and v could be viewed as constants. Note that the unique unknown in the objective function of the inner $\min_N$ problem is N. Let

$$F(N;x,y,w,v) \triangleq G(N). \tag{12}$$

Then the inner $\min_N$ problem of Eq.(10b) becomes $\min G(N)$, which is a local minima problem. From the second derivative test (SDT) for local extreme values, if $G'(N_0)=0$ and $G''(N_0)>0$, then $G(N)$ has a local minimum at $N=N_0$. Consequently, Eq.(10b) can be reformulated as the following traditional mathematical program:

$$\Psi_\alpha^{\mathrm{U}} = \min_{N_0>0} G(N), \quad \text{s.t. } x_\alpha^{\mathrm{L}}\le x\le x_\alpha^{\mathrm{U}},\ y_\alpha^{\mathrm{L}}\le y\le y_\alpha^{\mathrm{U}},\ w_\alpha^{\mathrm{L}}\le w\le w_\alpha^{\mathrm{U}},\ v_\alpha^{\mathrm{L}}\le v\le v_\alpha^{\mathrm{U}}. \tag{13}$$

At least one x, y, w or v must hit the boundary of their α-cuts to satisfy $\eta_{\tilde{\Psi}}=\alpha$. From a knowledge of calculus, a unique minimum and a unique maximum of the objective function in Eqs.(11) and (13) exist, respectively. The lower bound Ψ_α^{L} and upper bound Ψ_α^{U} of the α-cuts of $\tilde{\Psi}$ can be found by solving Eqs.(11) and (13), respectively, which involves the systematic study of how the optimal solutions change when $x_\alpha^{\mathrm{L}}, x_\alpha^{\mathrm{U}}, y_\alpha^{\mathrm{L}}, y_\alpha^{\mathrm{U}}, w_\alpha^{\mathrm{L}}, w_\alpha^{\mathrm{U}}, v_\alpha^{\mathrm{L}}$ and v_α^{U} vary over the interval $\alpha\in(0, 1]$. They fall into the category of parametric NLP (Gal, 1979).

The crisp interval $[\Psi_\alpha^{\mathrm{L}}, \Psi_\alpha^{\mathrm{U}}]$ solved from Eqs.(11) and (13) represents the α-cuts of $\eta_{\tilde{\Psi}}$. Again, by applying the results in (Kaufmann, 1975; Zimmermann, 2001) and convexity properties to $\tilde{\Psi}$, we have $\Psi_{\alpha_1}^{\mathrm{L}}\ge\Psi_{\alpha_2}^{\mathrm{L}}$ and $\Psi_{\alpha_1}^{\mathrm{U}}\le\Psi_{\alpha_2}^{\mathrm{U}}$, where $0<\alpha_2<\alpha_1<1$. In other words, Ψ_α^{L} is increasing with respect to α, and Ψ_α^{U} is decreasing with respect to α.

Consequently, the membership function $\eta_{\tilde{\Psi}}(z)$ can be found from Eqs.(11) and (13).

If both Ψ_α^{L} and Ψ_α^{U} of Eqs.(11) and (13) are invertible with respect to *α*, a left shape function $L(z)=(\Psi_\alpha^{\mathrm{L}})^{-1}$ and a right shape function $R(z)=(\Psi_\alpha^{\mathrm{U}})^{-1}$ can be obtained, from which the membership function $\eta_{\tilde{\Psi}}$ is constructed:

$$\eta_{\tilde{\Psi}}(z)=\begin{cases} L(z), & \Psi_{\alpha=0}^{\mathrm{L}} \le z < \Psi_{\alpha=1}^{\mathrm{L}}, \\ 1, & \Psi_{\alpha=1}^{\mathrm{L}} \le z \le \Psi_{\alpha=1}^{\mathrm{U}}, \\ R(z), & \Psi_{\alpha=1}^{\mathrm{U}} < z \le \Psi_{\alpha=0}^{\mathrm{U}}. \end{cases} \tag{14}$$

In most cases, the values of Ψ_α^{L} and Ψ_α^{U} cannot be solved analytically. Consequently, a closed-form membership function for $\eta_{\tilde{\Psi}}$ cannot be obtained. However, the numerical solutions for Ψ_α^{L} and Ψ_α^{U} at different possibility levels can be collected to approximate the shapes of $L(z)$ and $R(z)$. That is, the set of crisp intervals $\{[\Psi_\alpha^{\mathrm{L}}, \Psi_\alpha^{\mathrm{U}}] \mid 0 \le \alpha \le 1\}$ reveals the shape of $\eta_{\tilde{\Psi}}$.

Since the optimal threshold is described by a membership function, the value conserves completely all of the fuzziness of the arrival rate, service rate, holding cost, and setup cost. However, managers or practitioners would prefer one crisp value rather than a fuzzy set for the optimal threshold. In order to overcome this problem, we defuzzify the fuzzy optimal threshold by Yager (1986)'s ranking index method. Since Yager's ranking index method possesses the property of area compensation, we adopt this method for transforming the fuzzy optimal threshold into a crisp one to provide a suitable threshold value for the controllable queueing systems. The recommended optimum threshold value is calculated by

$$O(N^*)=\int_0^1 \frac{(N^*)_\alpha^{\mathrm{L}}+(N^*)_\alpha^{\mathrm{U}}}{2}\,\mathrm{d}\alpha. \tag{15a}$$

If $O(N^*)$ is not an integer, the best positive integer value of *N* is one of the integers surrounding $O(N^*)$. The corresponding minimum cost is

$$O(\Psi^*)=\int_0^1 \frac{\Psi_\alpha^{\mathrm{L}}+\Psi_\alpha^{\mathrm{U}}}{2}\mathrm{d}\alpha. \tag{15b}$$

NUMERICAL EXAMPLE

To demonstrate the practical use of the proposed approach, an example inspired by Arumuganathan and Jeyakumar (2005) is solved.

Example 1 A pump manufacturing company manufactures different types of pumps, which require shafts of various dimensions. The arrival of shafts from the turning center to the CNC (computer numerical control) copy turning center follows a Poisson process with a fuzzy arrival rate $\tilde{\lambda}$. After all the shafts are served in the queue exhaustively, the operator immediately shuts down the CNC machine. The operator uses this time for doing some other work such as making the templates for copy turning, checking the components. As soon as the quantity of arriving shafts reaches the minimum quantity *N*, the operator turns on the CNC machine and starts the copy turning process with a fuzzy service rate $\tilde{\mu}$ for each shaft. In order to utilize the CNC machine and the operator efficiently, the management wishes to determine the optimal threshold value (minimum quantity) that minimizes the total average cost. The problem can be modeled as a controllable queueing system with *N* policy.

Suppose that the arrival rate, service rate, holding cost and setup cost are trapezoidal fuzzy numbers represented by $\tilde{\lambda}=[1,2,3,4]$, $\tilde{\mu}=[5,6,7,8]$, $\tilde{C}_{\mathrm{h}}=[0.5,0.6,0.7,0.8]$ and $\tilde{C}_{\mathrm{s}}=[30,40,50,60]$, respectively. First, it is easy to find that $[x_\alpha^{\mathrm{L}}, x_\alpha^{\mathrm{U}}]=[\alpha+1, 4-\alpha]$, $[y_\alpha^{\mathrm{L}}, y_\alpha^{\mathrm{U}}]=[\alpha+5, 8-\alpha]$, $[w_\alpha^{\mathrm{L}}, w_\alpha^{\mathrm{U}}]=[0.1\alpha+0.5, 0.8-0.1\alpha]$ and $[v_\alpha^{\mathrm{L}}, v_\alpha^{\mathrm{U}}]=[10\alpha+30, 60-10\alpha]$. It is easy to find a minimum point N_0 such that $G'(N_0)=0$ and $G''(N_0)>0$ [$G(N)$ is as defined in Eq.(12)]; that is, $w/2-vx(y-x)/(yN_0^2)=0$ (or critical point $N_0=\sqrt{2vx(y-x)/(wy)}$) and $G''(N_0)=2vx(y-x)/(yN_0^3)>0$. The latter always holds since $y-x>0$ and $N_0>0$. Following Eqs.(11) and (13), the membership function of $\tilde{\Psi}$ can be formulated as

$$\Psi_\alpha^{\mathrm{L}}=\min G(N), \tag{16a}$$

s.t. $2vx(y-x)/(yN_0^3)>0$, $\alpha+1\le x\le 4-\alpha$, $\alpha+5\le y\le 8-\alpha$, $0.1\alpha+0.5\le w\le 0.8-0.1\alpha$, $10\alpha+30\le v\le 60-10\alpha$;

$$\Psi_\alpha^{\mathrm{U}} = \max G(N_0), \tag{16b}$$

s.t. $N_0 = \sqrt{2vx(y-x)/(wy)}$, $\alpha+1 \le x \le 4-\alpha$, $\alpha+5 \le y \le 8-\alpha$, $0.1\alpha+0.5 \le w \le 0.8-0.1\alpha$, $10\alpha+30 \le v \le 60-10\alpha$.

It is easy to find the objective values of Eq.(16) by using a parametric NLP solver. From Eq.(16), the solutions of the membership function for $\tilde{\Psi}$ in terms of α are obtained as follows:

$$\Psi_\alpha^{\mathrm{L}} = \frac{1}{40(\alpha+5)}\Big(\alpha^3 + 13\alpha^2 + 59\alpha + 95 + 80\left|\alpha+5\right|\sqrt{2(\alpha+1)(\alpha+3)}\Big),$$

which occurs at $(N^*)_\alpha^{\mathrm{L}} = 20[2(\alpha+1)(\alpha+3)(\alpha+5)^{-2}]^{1/2}$, and

$$\Psi_\alpha^{\mathrm{U}} = \frac{1}{40(\alpha-8)}\Big(\alpha^3 - 22\alpha^2 + 164\alpha - 416 - 80\left|\alpha-8\right|\sqrt{2(\alpha-4)(\alpha-6)}\Big)$$

occurring at $(N^*)_\alpha^{\mathrm{U}} = 20[2(\alpha-4)(\alpha-6)(\alpha-8)^{-2}]^{1/2}$.

With the help of Matlab 7.0.4, the inverse functions of Ψ_α^{L} and Ψ_α^{U} exist, yielding the membership function

$$\eta_{\tilde{\Psi}}(z) = \begin{cases} -4 + \dfrac{\sqrt{3}}{3}P - \dfrac{1}{3}\sqrt{D}, & \dfrac{19+80\sqrt{6}}{40} \le z < \dfrac{87}{10}, \\ 1, & \dfrac{87}{10} \le z \le \dfrac{39+80\sqrt{30}}{40}, \\ 7 - \dfrac{\sqrt{3}}{3}P + \dfrac{1}{3}\sqrt{D}, & \dfrac{39+80\sqrt{30}}{40} < z \le \dfrac{13+80\sqrt{3}}{10}, \end{cases}$$

where

$$D = 38\,382 + 240z - 115\,200\sqrt{3}/P - 3\sqrt[3]{Q} - (30\,604\,827 - 383\,280z - 4800z^2)/\sqrt[3]{Q},$$

$$Q = -617\,278\,920z - 32\,445\,267\,227 + 7\,665\,600z^2 + 64\,000z^3 + 19\,200(-24\,454\,521\,681 + 451\,731\,240z - 3\,025\,965\,900z^2 - 76\,536\,000z^3 - 480\,000z^4)^{1/2},$$

$$P = \left[\left(6397\sqrt[3]{Q} + 40\sqrt[3]{Q}z + \sqrt[3]{Q^2} + 10\,201\,609 + 127\,760z + 1600z^2\right)\Big/\sqrt[3]{Q}\right]^{1/2}.$$

The numerical results of the optimal solution $[\Psi_\alpha^{\mathrm{L}}, \Psi_\alpha^{\mathrm{U}}]$ of the membership function $\tilde{\Psi}$ at different possibility levels are shown in Fig.1. Crisp intervals for the fuzzy average total cost per unit time at different possibility levels α are given in Table 1.

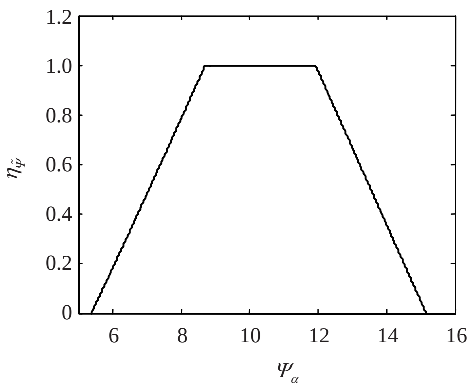

Fig.1 The membership function for the minimal average total cost per unit time

Table 1 The α-cuts of the arrival rate (x), service rate (y), holding cost (w), setup cost (v), optimum threshold (N^*), lower bound and upper bound of the membership function for the minimal average total cost per unit time (Ψ) at different possibility levels (α)

α	x_α^{L}	x_α^{U}	y_α^{L}	y_α^{U}	w_α^{L}	w_α^{U}	v_α^{L}	v_α^{U}	$(N^*)_\alpha^{\mathrm{L}}$	$(N^*)_\alpha^{\mathrm{U}}$	Ψ_α^{L}	Ψ_α^{U}
0	1.00	4.00	5.00	8.00	0.50	0.80	30.00	60.00	9.80	17.32	5.3740	15.1564
0.1	1.10	3.90	5.10	7.90	0.51	0.79	31.00	59.00	10.24	17.17	5.7183	14.8329
0.2	1.20	3.80	5.20	7.80	0.52	0.78	32.00	58.00	10.66	17.02	6.0586	14.5096
0.3	1.30	3.70	5.30	7.70	0.53	0.77	33.00	57.00	11.05	16.87	6.3956	14.1865
0.4	1.40	3.60	5.40	7.60	0.54	0.76	34.00	56.00	11.43	16.71	6.7299	13.8636
0.5	1.50	3.50	5.50	7.50	0.55	0.75	35.00	55.00	11.78	16.55	7.0620	13.5409
0.6	1.60	3.40	5.60	7.40	0.56	0.74	36.00	54.00	12.12	16.38	7.3922	13.2184
0.7	1.70	3.30	5.70	7.30	0.57	0.73	37.00	53.00	12.45	16.20	7.7209	12.8960
0.8	1.80	3.20	5.80	7.20	0.58	0.72	38.00	52.00	12.75	16.02	8.0483	12.5738
0.9	1.90	3.10	5.90	7.10	0.59	0.71	39.00	51.00	13.05	15.84	8.3746	12.2516
1.0	2.00	3.00	6.00	7.00	0.60	0.70	40.00	50.00	13.33	15.65	8.7000	11.9295

In Example 1, for the minimal average total cost $\tilde{\Psi}$, the range of $\tilde{\Psi}$ at α=1 is $[87/10, (39+80\sqrt{30})/40]$ (or approximately [8.7, 11.9295]), indicating that it is definitely possible that the minimal average total cost falls between 8.7 and 11.9295. Moreover, the range of $\tilde{\Psi}$ at α=0 is $[(19+80\sqrt{6})/40, (13+80\sqrt{3})/10]$ (or approximately [5.3740, 15.1564]), indicating that the minimal average total cost will never exceed 15.1564 or fall below 5.3740. In contrast, the optimal threshold value N^* has also been found. At the cut α=0 of $\tilde{\Psi}$, the lower bound $(\Psi)^{\mathrm{L}}_{\alpha=0}=(19+80\sqrt{6})/40$ occurs at $(N^*)^{\mathrm{L}}_{\alpha=0}=4\sqrt{6}$ (or approximately 10), and the upper bound $(\Psi)^{\mathrm{U}}_{\alpha=0}=(13+80\sqrt{3})/10$ occurs at $(N^*)^{\mathrm{U}}_{\alpha=0}=10\sqrt{3}$ (or approximately 17). At the other extreme end of α=1, the lower bound $(\Psi)^{\mathrm{L}}_{\alpha=1}=87/10$ occurs at $(N^*)^{\mathrm{L}}_{\alpha=1}=40/3$ (or approximately 13), and the upper bound $(\Psi)^{\mathrm{U}}_{\alpha=1}=(39+80\sqrt{30})/40$ occurs at $(N^*)^{\mathrm{U}}_{\alpha=1}=20\sqrt{30}/7$ (or approximately 16).

As this example demonstrates, the approach proposed in this study provides practical information for system designers and practitioners.

Applying Eq.(14), the recommended optimal threshold value and its corresponding minimum cost can be obtained, respectively, as

$$O(N^*)=\int_0^1 \frac{(N^*)^{\mathrm{L}}_{\alpha}+(N^*)^{\mathrm{U}}_{\alpha}}{2}\mathrm{d}\alpha$$
$$=10\int_0^1\left(\sqrt{\frac{2(\alpha+1)(\alpha+3)}{(\alpha+5)^2}}+\sqrt{\frac{2(\alpha-4)(\alpha-6)}{(\alpha-8)^2}}\right)\mathrm{d}\alpha$$
$$=14.1187\approx 14,$$

and

$$O(\Psi^*)=\int_0^1 \frac{\Psi^{\mathrm{L}}_{\alpha}+\Psi^{\mathrm{U}}_{\alpha}}{2}\mathrm{d}\alpha\approx 10.2978.$$

Comparison analysis results using fuzzy theory and conventional method

As shown in the example, the recorded values of the parameters are approximated to the constants. It is not proper to analyze the characters of the system with a single crisp mean value. Traditionally, engineers use the mean of these possible observations for the parameters as the estimates for finding system performance measures such as the mean system size, the mean busy cycle, the minimum cost function, and the optimal operation policy (threshold) of the system. In the above example, if the traditional approach is used to find the system performance measures, the arrival rate, service rate, holding cost and setup cost are estimated as $\bar{\lambda}=2.5$, $\bar{\mu}=6.5$, $\bar{C}_{\mathrm{h}}=0.65$ and $\bar{C}_{\mathrm{s}}=45$, respectively; the minimum cost and the optimal operation threshold will be $G(N^*)$=10.4681 and N^*=15, respectively (see Appendix). However, if the fuzzy parameters are used to find the minimum cost (the optimal operation threshold), the ranges are between 5.3740 and 15.1564 (between 9.80 and 17.32). Table 1 illustrates 10 crisp intervals for fuzzy minimum cost and optimal threshold at different possibility levels. Although the possibility of some occurrences is very low, the system is still affected by these occurrences. Therefore, the occurrences cannot be neglected. Considering that engineers prefer a suitable single value threshold for practical use with these approximate parameters, this study uses an approach following Yager's ranking index method. Based on the illustrated example, the minimum cost (the suitable operation threshold) is 10.2978 (Eq.(15b)), lower than 10.4681 (Eq.(13)), the values derived from the traditional approach. Therefore, the risk of over-estimation may happen when we adopt the traditional approach.

CONCLUSION

This paper applies the concepts of α-cuts and Zadeh's extension principle to a controllable queueing system and constructs membership functions of the minimal average total cost per unit time using paired NLP models. Following the proposed approach, α-cuts of the membership functions are found and their interval limits are inverted to attain explicit closed-form expressions for the minimal average total cost per unit time. Even when the membership function intervals cannot be inverted, system managers or designers can perform numerical experiments to examine the corresponding α-cuts and then use this information to develop or improve system processes.

Note that the arrival rate, service rate, holding cost and setup cost are assumed to be trapezoidal fuzzy numbers in the numerical example. Clearly, the proposed approach is not confined to trapezoidal fuzzy numbers. It can be applied to other cases with related system parameters involving convex fuzzy sets such as LR-fuzzy numbers.

References

Arumuganathan, R., Jeyakumar, S., 2005. Steady state analysis of a bulk queue with multiple vacations, setup times with *N*-policy and closedown times. *Appl. Math. Model.*, **29**(10):972-986. [doi:10.1016/j.apm.2005.02.013]

Bell, C.E., 1971. Characterization and computation of optimal policies for operating an *M*/*G*/1 queueing system with removable server. *Oper. Res.*, **19**(1):208-218. [doi:10.1287/opre.19.1.208]

Buckley, J.J., Feuring, T., Hayashi, Y., 2001. Fuzzy queueing theory revisited. *Int. J. Uncert., Fuzz., Knowl.-Based Syst.*, **9**(5):527-537. [doi:10.1142/S0218488501001046]

Buzacott, J., Shanthikumar, J., 1993. Stochastic Models of Manufacturing Systems. Prentice-Hall, Englewood Cliffs, NJ.

Choudhury, G., Madan, K.C., 2005. A two-stage batch arrival queueing system with a modified Bernoulli schedule vacation under *N*-policy. *Math. Comput. Model.*, **42**(1-2):71-85. [doi:10.1016/j.mcm.2005.04.003]

Choudhury, G., Paul, M., 2006. A batch arrival queue with a second optional service channel under *N*-policy. *Stoch. Anal. Appl.*, **24**(1):1-21. [doi:10.1080/07362990500397277]

Gal, T., 1979. Postoptimal Analysis, Parametric Programming, and Related Topics. McGraw-Hill, New York.

Gross, D., Harris, C.M., 1998. Fundamentals of Queueing Theory (3rd Ed.). John Wiley, New York.

Heyman, D.P., 1968. Optimal operating policies for *M*/*G*/1 queueing system. *Oper. Res.*, **16**(2):362-382. [doi:10.1287/opre.16.2.362]

Hillier, F.S., Lieberman, G.J., 2001. Introduction to Operations Research (7th Ed.). McGraw-Hill, Singapore.

Kaufmann, A., 1975. Introduction to the Theory of Fuzzy Subsets, Volume 1. Academic Press, New York.

Kella, O., 1989. The threshold policy in the *M*/*G*/1 queue with server vacations. *Nav. Res. Logist.*, **36**(1):111-123. [doi:10.1002/1520-6750(198902)36:1<111::AID-NAV3220360109>3.0.CO;2-3]

Kleinrock, L., 1975. Queueing Systems, Vol. 1: Theory. Wiley, New York.

Lee, H.S., Srinivasan, M.M., 1989. Control policies for the $M^{[x]}/G/1$ queueing system. *Manag. Sci.*, **35**(6):708-721. [doi:10.1287/mnsc.35.6.708]

Lee, H.W., Park, J.O., 1997. Optimal strategy in *N*-policy production system with early set-up. *J. Oper. Res. Soc.*, **48**(3):306-313. [doi:10.1057/palgrave.jors.2600354]

Lee, H.W., Lee, S.S., Chae, K.C., 1994a. Operating characteristics of $M^X/G/1$ queue with *N* policy. *Queueing Systems*, **15**(1-4):387-399. [doi:10.1007/BF01189247]

Lee, H.W., Lee, S.S., Park, J.O., Chae, K.C., 1994b. Analysis of $M^{[x]}/G/1$ queue with *N* policy and multiple vacations. *J. Appl. Probab.*, **31**(2):467-496. [doi:10.2307/3215040]

Lee, S.S., Lee, H.W., Yoon, S.H., Chae, K.C., 1995. Batch arrival queue with *N*-policy and single vacation. *Comput. Oper. Res.*, **22**(2):173-189. [doi:10.1016/0305-0548(94)E0015-Y]

Li, R.J., Lee, E.S., 1989. Analysis of fuzzy queues. *Comput. Math. Appl.*, **17**(7):1143-1147. [doi:10.1016/0898-1221(89)90044-8]

Pearn, W.L., Ke, J.C., Chang, Y.C., 2004. Sensitivity analysis of the optimal management policy for a queueing system with a removable and non-reliable server. *Comput. Ind. Eng.*, **46**(1):87-99. [doi:10.1016/j.cie.2003.11.001]

Tadj, L., Choudhury, G., 2005. Optimal design and control of queues. *TOP*, **13**(2):359-414. [doi:10.1007/BF02579061]

Tadj, L., Choudhury, G., Tadj, C., 2006a. A quorum queueing system with a random setup time under *N*-policy with Bernoulli vacation schedule. *Qual. Technol. Quantit. Manag.*, **3**(2):145-160.

Tadj, L., Choudhury, G., Tadj, C., 2006b. A bulk quorum queueing system with a random setup time under *N*-policy with Bernoulli vacation schedule. *Stoch.: Int. J. Probab. Stoch. Processes*, **78**(1):1-11. [doi:10.1080/17442500500397574]

Taha, H.A., 2003. Operations Research: An Introduction (7th Ed.). Prentice-Hall, New Jersey.

Wang, K.H., Kao, H.T., Chen, G., 2004. Optimal management of a removable and non-reliable server in an infinite and a finite $M/H_k/1$ queueing system. *Int. J. Qual. Technol. Quantit. Manag.*, **1**(2):325-339.

Yadin, M., Naor, P., 1963. Queueing systems with a removable service station. *J. Oper. Res. Soc.*, **14**(4):393-405. [doi:10.1057/jors.1963.63]

Yager, R.R., 1986. A characterization of the extension principle. *Fuzzy Sets Syst.*, **18**(3):205-217. [doi:10.1016/0165-0114(86)90002-3]

Zadeh, L.A., 1978. Fuzzy sets as a basis for a theory of possibility. *Fuzzy Sets Syst.*, **1**(1):3-28. [doi:10.1016/0165-0114(78)90029-5]

Zhang, B., 2006. A Fuzzy-logic-based Methodology for Batch Process Scheduling. IEEE Systems and Information Engineering Design Symp., p.101-105. [doi:10.1109/SIEDS.2006.278721]

Zhang, R., Phillis, Y.A., Kouikoglou, V.S., 2005. Fuzzy Control of Queueing Systems. Springer, New York.

Zimmermann, H.J., 2001. Fuzzy Set Theory and Its Applications (4th Ed.). Kluwer Academic, Boston.

APPENDIX: MINIMUM COST AND OPTIMAL OPERATION THRESHOLD OF EQ.(1)

$$G(N)=C_{\mathrm{h}}\left(\frac{N-1}{2}+\frac{\lambda}{\mu}+\frac{\mu}{\mu-\lambda}\right)+\frac{C_{\mathrm{s}}\lambda(\mu-\lambda)}{N\mu},\quad \text{(A1)}$$

$$\frac{\mathrm{d}G(N)}{\mathrm{d}N}=\frac{C_{\mathrm{h}}}{2}-\frac{C_{\mathrm{s}}\lambda(\mu-\lambda)}{N^2\mu},$$

$$\frac{\mathrm{d}^2G(N)}{\mathrm{d}N^2}=\frac{2C_{\mathrm{s}}\lambda(\mu-\lambda)}{N^3\mu}>0,$$

where μ, λ, C_{s}, C_{h} and N are positive, and $\mu>\lambda$. Let $\frac{\mathrm{d}G(N)}{\mathrm{d}N}=0$, then

$$N^*=\sqrt{\frac{2C_{\mathrm{s}}\lambda(\mu-\lambda)}{C_{\mathrm{h}}\mu}}.\quad \text{(A2)}$$