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An implicit symmetry constraint of the modified Korteweg-de Vries (mKdV) equation

Ying YOU[†1], Jing YU[†‡2], Qiao-yun JIANG[3]

([1]*Department of Mathematics, Zhejiang University, Hangzhou 310027, China*)

([2]*School of Science, Hangzhou Dianzi University, Hangzhou 310018, China*)

([3]*College of Science, Nantong University, Nantong 226007, China*)

[†]E-mail: youying1024@126.com; yujing615@126.com

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Abstract: In this paper, an implicit symmetry constraint is calculated and its associated binary nonlinearization of the Lax pairs and the adjoint Lax pairs is carried out for the modified Korteweg-de Vries (mKdV) equation. After introducing two new independent variables, we find that under the implicit symmetry constraint, the spatial part and the temporal part of the mKdV equation are decomposed into two finite-dimensional systems. Furthermore we prove that the obtained finite-dimensional systems are Hamiltonian systems and completely integrable in the Liouville sense.

Key words: Implicit symmetry constraint, Completely integrable Hamiltonian system, Modified Korteweg-de Vries (mKdV) equation

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INTRODUCTION

It is well known that the ZS-AKNS spectral problem is given by

$$\boldsymbol{\varphi}_x = \begin{pmatrix} -\lambda & q \\ r & \lambda \end{pmatrix} \boldsymbol{\varphi}, \quad \boldsymbol{\varphi} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \tag{1}$$

where λ is a spectral parameter, q and r are potentials, and $\boldsymbol{\varphi}$ is an eigenfunction. By imposing the reduction condition $r=-q$ in Eq.(1), we obtain the famous modified Korteweg-de Vries (mKdV) equation, which has been widely studied (Gesztesy *et al*., 1991; Zhou and Ma, 1998; Geng and Cao, 1999; Li and Ma, 2002; Yan, 2002; Zeng *et al*., 2002; Zhang, 2002; He and Chen, 2005; Wu *et al*., 2007).

On the other hand, the method of nonlinearization of Lax pairs for classic integrable systems has been widely researched in the past several years. Cao (1988) proposed the mono nonliearization, and Ma and Strampp (1994) proposed the binary nonlinearization. Many successful examples have been nonlinearized into finite-dimensional Liouville integrable Hamiltonian systems (Zeng and Li, 1989; 1990; Cao and Geng, 1990; 1992; Ma, 1995a; 1997; Ma *et al*., 1996; Zhou, 1998; Ma and Zhou, 2001; Yu and Zhou, 2006). These examples show that what they studied is an explicit constraint, i.e., the potential can be expressed by an eigenfunction explicitly. Li and Ma (2000) considered the binary nonlinearization of the AKNS spectral problem under a higher-order symmetry constraint, i.e., the potential can be expressed by an eigenfunction implicitly, which makes the nonlinearization procedure much more complex. Ma (1995b) considered an explicit symmetry constraint for the mKdV equation by binary nonlinearization. It is natural to raise the following question: is the mKdV equation nonlinearized into finite-dimensional integrable Hamiltonian systems under a higher-order symmetry constraint? This paper will give an affirmative answer to it.

[‡] Corresponding author

The organization of the paper is as follows. In the next section, the mKdV hierarchy is briefly recalled from a spectral problem. In Section 3, we will give an implicit symmetry constraint between the potential of the mKdV equation and the eigenfunction of the corresponding spectral problems. And in Section 4, after introducing two new independent variables, the mKdV equation is decomposed into two finite-dimensional systems, and furthermore it is proved that the obtained systems are Hamiltonian systems, and the functionally independent and involutive integral of motion are given explicitly. The conclusions and remarks are given in Section 5.

mKdV HIERARCHY

Let us start with the following spectral problem:

$$\boldsymbol{\phi}_x = \boldsymbol{U}\boldsymbol{\phi},\ \ \boldsymbol{U} = \boldsymbol{U}(u,\lambda) = \begin{pmatrix} u & \lambda \\ -1 & -u \end{pmatrix},\ \ \boldsymbol{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad (2)$$

where λ is a spectral parameter and $u=u(x, t)$ is a potential.

In what follows, we would derive the mKdV hierarchy. Take

$$\boldsymbol{V} = \begin{pmatrix} a & b\lambda \\ c & -a \end{pmatrix} = \sum_{j=0}^{\infty} \begin{pmatrix} a_j & b_j\lambda \\ c_j & -a_j \end{pmatrix} \lambda^{-j},$$

then the adjoint representation equation (Tu, 1989)

$$\boldsymbol{V}_x = [\boldsymbol{U}\ \ \boldsymbol{V}] = \boldsymbol{U}\boldsymbol{V} - \boldsymbol{V}\boldsymbol{U} \quad (3)$$

gives

$$\begin{cases} b_0 + c_0 = 0, \\ a_{i,x} = b_{i+1} + c_{i+1}, \\ b_{i,x} = 2ub_i - 2a_i, \\ c_{i,x} = -2a_i - 2uc_i, \end{cases} \quad i \ge 0, \quad (4)$$

which results in a recursion relation to determine a_i, b_i, c_i:

$$\begin{cases} a_{i+1} = \mathfrak{L}a_i,\ \mathfrak{L} = -\dfrac{1}{4}\partial^2 + u\partial^{-1}u\partial, \\ b_{i+1} = \dfrac{1}{2}a_{i,x} + \partial^{-1}u\partial a_i, \\ c_{i+1} = \dfrac{1}{2}a_{i,x} - \partial^{-1}u\partial a_i, \end{cases} \quad i \ge 0, \quad (5)$$

If we choose the initial value

$$a_0=u,\ \ b_0=1,\ \ c_0=-1, \quad (6)$$

and further assume

$$a_i\big|_{u=0} = b_i\big|_{u=0} = c_i\big|_{u=0} = 0,\ \ i \ge 1,$$

which means setting all constants of integration to be zero, then all a_i, b_i, c_i ($i\ge1$) are uniquely worked out by the recursion relation Eq.(5). For example, the first terms are

$$a_1 = -\frac{1}{4}u_{xx} + \frac{1}{2}u^3,\ \ b_1 = \frac{1}{2}u_x + \frac{1}{2}u^2,\ c_1 = \frac{1}{2}u_x - \frac{1}{2}u^2,$$

$$a_2 = \frac{1}{16}u_{xxxx} - \frac{5}{8}u^2u_{xx} - \frac{5}{8}uu_x^2 + \frac{3}{8}u^5,$$

$$b_2 = -\frac{1}{8}u_{xxx} + \frac{3}{4}u^2u_x - \frac{1}{4}uu_{xx} + \frac{1}{8}u_x^2 + \frac{3}{8}u^4,$$

$$c_2 = -\frac{1}{8}u_{xxx} + \frac{3}{4}u^2u_x + \frac{1}{4}uu_{xx} - \frac{1}{8}u_x^2 - \frac{3}{8}u^4.$$

Now associate the mKdV spectral problem Eq.(2) with the following auxiliary problem:

$$\boldsymbol{\phi}_{t_n} = \boldsymbol{V}^{(n)}\boldsymbol{\phi} = \boldsymbol{V}^{(n)}(u,\lambda)\boldsymbol{\phi}, \quad (7)$$

with

$$\boldsymbol{V}^{(n)} = (\lambda^n\boldsymbol{V})_+ - \begin{pmatrix} 0 & b_{n+1} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} (a\lambda^n)_+ & (b\lambda^n)_+\lambda \\ (c\lambda^n)_+ & -(a\lambda^n)_+ \end{pmatrix},\ \ n \ge 0, \quad (8)$$

where the subscript '+' denotes taking the nonnegative power of λ. The compatible conditions of the spectral problem Eq.(2) and the auxiliary problem Eq.(7) determine zero curvature equations

$$\boldsymbol{U}_{t_n} - \boldsymbol{V}_x^{(n)} + [\boldsymbol{U}\ \ \boldsymbol{V}^{(n)}] = \boldsymbol{0},\ \ n \ge 1,$$

which lead to isospectral ($\lambda_{t_n} = 0$) integrable hierarchy of the mKdV equations

$$u_{t_n} = a_{n,x},\ \ n \ge 1. \quad (9)$$

Using the trace identity in (Tu, 1989), we find that Eq.(9) can be written in the form of a Hamiltonian system

$$u_{t_n} = \partial\frac{\delta\tilde{H}_n}{\delta u},\ \ \frac{\delta\tilde{H}_n}{\delta u} = a_n, \quad (10)$$

where the Hamiltonian functions are defined by

$$\tilde{H}_n = \int \frac{b_{n+1} - c_{n+1}}{2(2n+1)} \mathrm{d}x.$$

For n=1, we obtain the first nonlinear system in the mKdV integrable hierarchy Eq.(9)

$$u_{t_1} = -\frac{1}{4}u_{xxx} + \frac{3}{2}u^2 u_x, \quad (11)$$

which is exactly the mKdV equation, whose auxiliary problem is given by

$$\boldsymbol{\phi}_{t_1} = \boldsymbol{V}^{(1)}\boldsymbol{\phi}, \quad (12)$$

where

$$\boldsymbol{V}^{(1)} = \begin{pmatrix} u\lambda - \frac{1}{4}u_{xx} + \frac{1}{2}u^3 & \lambda^2 + \frac{1}{2}(u_x + u^2)\lambda \\ -\lambda + \frac{1}{2}(u_x - u^2) & -u\lambda + \frac{1}{4}u_{xx} - \frac{1}{2}u^3 \end{pmatrix}.$$

AN IMPLICIT SYMMETRY CONSTRAINT

Consider the adjoint spectral problem of the spectral problem Eq.(2):

$$\boldsymbol{\psi}_x = -\boldsymbol{U}^{\mathrm{T}}\boldsymbol{\psi} = \begin{pmatrix} -u & 1 \\ -\lambda & u \end{pmatrix}\boldsymbol{\psi}, \ \boldsymbol{\psi} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \quad (13)$$

From (Ma *et al.*, 1996), it is not difficult to find that

$$\frac{\delta\lambda}{\delta u} = \frac{1}{E}(\phi_1\psi_1 - \phi_2\psi_2), \quad (14)$$

where $E = -\int \phi_2\psi_1 \mathrm{d}x$. Making use of Eq.(5) and imposing zero boundary conditions

$$\lim_{|x|\to+\infty} \boldsymbol{\phi} = \lim_{|x|\to+\infty} \boldsymbol{\psi} = \boldsymbol{0},$$

we can verify a simple characteristic property of the variational derivative

$$\mathfrak{L}\frac{\delta\lambda}{\delta u} = \lambda\frac{\delta\lambda}{\delta u}, \quad (15)$$

where $\mathfrak{L}$ and $\delta\lambda/\delta u$ are given by Eqs.(5) and (15), respectively.

Choosing N distinct parameters $\lambda_1, \lambda_2, \cdots, \lambda_N$, the spectral problem Eq.(2) and the adjoint spectral problem Eq.(13) become

$$\begin{pmatrix} \phi_{1j} \\ \phi_{2j} \end{pmatrix}_x = \begin{pmatrix} u & \lambda_j \\ -1 & -u \end{pmatrix}\begin{pmatrix} \phi_{1j} \\ \phi_{2j} \end{pmatrix} \quad (16a)$$

and

$$\begin{pmatrix} \psi_{1j} \\ \psi_{2j} \end{pmatrix}_x = \begin{pmatrix} -u & 1 \\ -\lambda_j & u \end{pmatrix}\begin{pmatrix} \psi_{1j} \\ \psi_{2j} \end{pmatrix}, \quad (16b)$$

respectively, where $1\le j\le N$.

To carry out the nonlinearization of the spectral problems, the traditional method is to make the following constraint

$$\frac{\delta\tilde{H}_k}{\delta u} = \sum_{j=1}^{N}\gamma_j\frac{\delta\lambda_j}{\delta u}, \quad k\ge 0, \quad (17)$$

where $\lambda_1, \lambda_2, \cdots, \lambda_N$ are constants.

When k=0 and γ_i=1/2 ($1\le i\le N$), we have $u=(<\boldsymbol{\Psi}_1, \boldsymbol{\Phi}_1>-<\boldsymbol{\Psi}_2, \boldsymbol{\Phi}_2>)/2$, whose constrained systems have been obtained in (Ma, 1995b). Here we take k=1 and γ_i=1/2 ($1\le i\le N$), and have

$$\frac{\delta\tilde{H}_1}{\delta u} = \frac{1}{2}\sum_{j=1}^{N}\frac{\delta\lambda_j}{\delta u} = \frac{1}{2}(<\boldsymbol{\Psi}_1,\boldsymbol{\Phi}_1> - <\boldsymbol{\Psi}_2,\boldsymbol{\Phi}_2>), \quad (18)$$

that is,

$$-\frac{1}{4}u_{xx} + \frac{1}{2}u^3 = \frac{1}{2}(<\boldsymbol{\Psi}_1,\boldsymbol{\Phi}_1> - <\boldsymbol{\Psi}_2,\boldsymbol{\Phi}_2>), \quad (19)$$

where $\boldsymbol{\Phi}_i = [\phi_{i1}, \phi_{i2}, ..., \phi_{iN}]^{\mathrm{T}}$, $\boldsymbol{\Psi}_i = [\psi_{i1}, \psi_{i2}, ..., \psi_{iN}]^{\mathrm{T}}$ (i=1, 2) and $<\cdot,\cdot>$ denotes the inner product in $\mathbb{R}^N$. We found that the constraint Eq.(19) is an implicit one, i.e., the potential u cannot be expressed by the eigenfunctions $\boldsymbol{\Psi}_i$ and $\boldsymbol{\Phi}_i$ (i=1, 2) explicitly.

BINARY NONLINEARIZATION

In this section, we will perform binary nonlinearization of the mKdV equation under an implicit symmetry constraint Eq.(19). We introduce two new independent variables

$$\phi_{N+1} = u, \quad \psi_{N+1} = \phi_{N+1}/2. \quad (20)$$

Substituting the constraint Eq.(19) and the two new variables in Eq.(20) into system Eq.(16), we obtain the following constrained system (1≤j≤N):

$$\begin{cases}\phi_{1j,x}=\phi_{N+1}\phi_{1j}+\lambda_j\phi_{2j},\ \phi_{2j,x}=-\phi_{1j}-\phi_{N+1}\phi_{2j},\\ \phi_{N+1,x}=2\psi_{N+1},\\ \psi_{1j,x}=-\phi_{N+1}\psi_{1j}+\psi_{2j},\ \psi_{2j,x}=-\lambda_j\psi_{1j}+\phi_{N+1}\psi_{2j},\\ \psi_{N+1,x}=-<\boldsymbol{\Psi}_1,\boldsymbol{\Phi}_1>+<\boldsymbol{\Psi}_2,\boldsymbol{\Phi}_2>+\phi_{N+1}^3,\end{cases} \tag{21}$$

which can be written as the following Hamiltonian form:

$$\boldsymbol{\Phi}_{i,x}=\frac{\partial H_0}{\partial\boldsymbol{\Psi}_i},\quad \boldsymbol{\Psi}_{i,x}=-\frac{\partial H_0}{\partial\boldsymbol{\Phi}_i},\quad i=1,2, \tag{22}$$

where

$$\begin{aligned}H_0=&\phi_{N+1}(<\boldsymbol{\Psi}_1,\boldsymbol{\Phi}_1>-<\boldsymbol{\Psi}_2,\boldsymbol{\Phi}_2>)+<\boldsymbol{\Lambda\Psi}_1,\boldsymbol{\Phi}_2>\\ &-<\boldsymbol{\Psi}_2,\boldsymbol{\Phi}_1>+\psi_{N+1}^2-\phi_{N+1}^4/4,\end{aligned}$$

with $\boldsymbol{\Lambda}$=diag{λ_1, λ_2, ..., λ_n}.

For t_1 part, we have N copies of the spectral problem and the adjoint spectral problem:

$$\begin{cases}\begin{pmatrix}\phi_{1j}\\ \phi_{2j}\end{pmatrix}_{t_1}=\begin{pmatrix}u\lambda_j-\frac{1}{4}u_{xx}+\frac{1}{2}u^3 & \lambda_j^2+\frac{1}{2}(u_x+u^2)\lambda_j\\ -\lambda_j+\frac{1}{2}(u_x-u^2) & -u\lambda_j+\frac{1}{4}u_{xx}-\frac{1}{2}u^3\end{pmatrix}\begin{pmatrix}\phi_{1j}\\ \phi_{2j}\end{pmatrix},\\ \begin{pmatrix}\psi_{1j}\\ \psi_{2j}\end{pmatrix}_{t_1}=\begin{pmatrix}-u\lambda_j+\frac{1}{4}u_{xx}-\frac{1}{2}u^3 & \lambda_j+\frac{1}{2}(u_x-u^2)\\ -\lambda_j^2-\frac{1}{2}(u_x+u^2)\lambda_j & u\lambda_j-\frac{1}{4}u_{xx}+\frac{1}{2}u^3\end{pmatrix}\begin{pmatrix}\psi_{1j}\\ \psi_{2j}\end{pmatrix},\end{cases}$$

$$1\le j\le N. \tag{23}$$

When the implicit symmetry constraint Eq.(19), the new variables in Eq.(20), and the system Eq.(21) are considered, system Eq.(23) is nonlinearized as follows:

$$\begin{aligned}\phi_{1j,t_1}=&\left(\phi_{N+1}\lambda_j+(<\boldsymbol{\Psi}_1,\boldsymbol{\Phi}_1>-<\boldsymbol{\Psi}_2,\boldsymbol{\Phi}_2>)/2\right)\boldsymbol{\Phi}_1\\ &+\left(\lambda_j^2+(\psi_{N+1}+\phi_{N+1}^2/2)\lambda_j\right)\phi_{2j},\\ \phi_{2j,t_1}=&\left(-\lambda_j+\psi_{N+1}-\phi_{N+1}^2/2\right)\phi_{1j}+\left(-\phi_{N+1}\lambda_j\right.\\ &\left.-(<\boldsymbol{\Psi}_1,\boldsymbol{\Phi}_1>-<\boldsymbol{\Psi}_2,\boldsymbol{\Phi}_2>)/2\right)\phi_{2j},\\ \phi_{N+1,t_1}=&<\boldsymbol{\Lambda\Psi}_1,\boldsymbol{\Phi}_2>+<\boldsymbol{\Psi}_2,\boldsymbol{\Phi}_1>,\\ \psi_{1j,t_1}=&\left(-\phi_{N+1}\lambda_j-(<\boldsymbol{\Psi}_1,\boldsymbol{\Phi}_1>-<\boldsymbol{\Psi}_2,\boldsymbol{\Phi}_2>)/2\right)\psi_{1j}\\ &+\left(\lambda_j-\psi_{N+1}+\phi_{N+1}^2/2\right)\psi_{2j},\\ \psi_{2j,t_1}=&\left(-\lambda_j^2-(\psi_{N+1}+\phi_{N+1}^2/2)\lambda_j\right)\psi_{1j}\\ &+\left(\phi_{N+1}\lambda_j+(<\boldsymbol{\Psi}_1,\boldsymbol{\Phi}_1>-<\boldsymbol{\Psi}_2,\boldsymbol{\Phi}_2>)/2\right)\psi_{2j},\\ \psi_{N+1,t_1}=&-<\boldsymbol{\Lambda\Psi}_1,\boldsymbol{\Phi}_1>+<\boldsymbol{\Lambda\Psi}_2,\boldsymbol{\Phi}_2>\\ &+\phi_{N+1}<\boldsymbol{\Lambda\Psi}_1,\boldsymbol{\Phi}_2>+\phi_{N+1}<\boldsymbol{\Psi}_2,\boldsymbol{\Phi}_1>,\end{aligned} \tag{24}$$

where 1≤j≤N. After a direct calculation, the nonlinearized system Eq.(24) is rewritten by

$$\boldsymbol{\Phi}_{i,t_1}=\frac{\partial H_1}{\partial\boldsymbol{\Psi}_i},\quad \boldsymbol{\Psi}_{i,t_1}=-\frac{\partial H_1}{\partial\boldsymbol{\Phi}_i},\quad i=1,2, \tag{25}$$

where

$$\begin{aligned}H_1=&<\boldsymbol{\Lambda}^2\boldsymbol{\Psi}_1,\boldsymbol{\Phi}_2>-<\boldsymbol{\Lambda\Psi}_2,\boldsymbol{\Phi}_1>+\phi_{N+1}(<\boldsymbol{\Lambda\Psi}_1,\boldsymbol{\Phi}_1>\\ &-<\boldsymbol{\Lambda\Psi}_2,\boldsymbol{\Phi}_2>)+(\psi_{N+1}+\phi_{N+1}^2/2)<\boldsymbol{\Lambda\Psi}_1,\boldsymbol{\Phi}_2>\\ &-(\psi_{N+1}-\phi_{N+1}^2/2)<\boldsymbol{\Psi}_2,\boldsymbol{\Phi}_1>+(<\boldsymbol{\Psi}_1,\boldsymbol{\Phi}_1>\\ &-<\boldsymbol{\Psi}_2,\boldsymbol{\Phi}_2>)^2/4.\end{aligned}$$

From Eq.(25), we know that the nonlinearized system Eq.(24) is a Hamiltonian system. For (4N+2)-dimensional Hamiltonian systems Eqs.(22) and (25), we need to find 2N+1 independent and involutive integrals of motion. To this end, we use the property Eq.(15) and the recursion relation Eq.(5), which ensure that

$$\begin{cases}\tilde{a}_i=\mathfrak{L}^{i-1}\tilde{a}_1=(<\boldsymbol{\Lambda}^{i-1}\boldsymbol{\Psi}_1,\boldsymbol{\Phi}_1>\\ \quad -<\boldsymbol{\Lambda}^{i-1}\boldsymbol{\Psi}_2,\boldsymbol{\Phi}_2>)/2,\quad i\ge1;\\ \tilde{b}_i=<\boldsymbol{\Lambda}^{i-2}\boldsymbol{\Psi}_2,\boldsymbol{\Phi}_1>,\qquad i\ge2;\\ \tilde{c}_i=<\boldsymbol{\Lambda}^{i-1}\boldsymbol{\Psi}_1,\boldsymbol{\Phi}_2>,\qquad i\ge1,\end{cases} \tag{26}$$

where $\tilde{P}_i$ denotes $P_i(u)$ (P=a, b, c) under the implicit symmetry constraint Eq.(19). At this time, the adjoint representation equation $\tilde{\boldsymbol{V}}_x=[\tilde{\boldsymbol{U}}\ \tilde{\boldsymbol{V}}]$ remains true. Thus an obvious equality $(\tilde{\boldsymbol{V}}^2)_x=[\tilde{\boldsymbol{U}}\ \tilde{\boldsymbol{V}}^2]$ yields

$$F_x=\left(\frac{1}{2}\mathrm{tr}\,(\tilde{\boldsymbol{V}}^2)\right)_x=\frac{\mathrm{d}}{\mathrm{d}x}(\tilde{a}^2+\tilde{b}\tilde{c}\lambda)=0,$$

which shows that F is a generating function of integrals of motion for Eq.(22). Let $F=\sum_{m\ge-1}F_m\lambda^{-m}$, we obtain the following formulas of integrals of motion:

$$F_0 = <\boldsymbol{\Psi}_1,\boldsymbol{\Phi}_2> -\psi_{N+1} + \phi_{N+1}^2/2,$$
$$F_1 = \phi_{N+1}(<\boldsymbol{\Psi}_1,\boldsymbol{\Phi}_1> - <\boldsymbol{\Psi}_2,\boldsymbol{\Phi}_2>) + <\boldsymbol{\Lambda\Psi}_1,\boldsymbol{\Phi}_2>$$
$$- <\boldsymbol{\Psi}_2,\boldsymbol{\Phi}_1> + (\psi_{N+1}+\phi_{N+1}^2/2)<\boldsymbol{\Psi}_1,\boldsymbol{\Phi}_2>,$$
$$F_m = \sum_{i=1}^{m-1}\Big((<\boldsymbol{\Lambda}^{i-1}\boldsymbol{\Psi}_1,\boldsymbol{\Phi}_1> - <\boldsymbol{\Lambda}^{i-1}\boldsymbol{\Psi}_2,\boldsymbol{\Phi}_2>)/4$$
$$\cdot(<\boldsymbol{\Lambda}^{m-i-1}\boldsymbol{\Psi}_1,\boldsymbol{\Phi}_1> - <\boldsymbol{\Lambda}^{m-i-1}\boldsymbol{\Psi}_2,\boldsymbol{\Phi}_2>)\Big)$$
$$+\sum_{i=2}^{m}\Big(<\boldsymbol{\Lambda}^{i-2}\boldsymbol{\Psi}_2,\boldsymbol{\Phi}_1><\boldsymbol{\Lambda}^{m-i}\boldsymbol{\Psi}_1,\boldsymbol{\Phi}_2>\Big)$$
$$+\phi_{N+1}(<\boldsymbol{\Lambda}^{m-1}\boldsymbol{\Psi}_1,\boldsymbol{\Phi}_1> - <\boldsymbol{\Lambda}^{m-1}\boldsymbol{\Psi}_2,\boldsymbol{\Phi}_2>)$$
$$+<\boldsymbol{\Lambda}^{m}\boldsymbol{\Psi}_1,\boldsymbol{\Phi}_2> + (\psi_{N+1}+\phi_{N+1}^2/2)<\boldsymbol{\Lambda}^{m-1}\boldsymbol{\Psi}_1,\boldsymbol{\Phi}_2>$$
$$-<\boldsymbol{\Lambda}^{m-1}\boldsymbol{\Psi}_2,\boldsymbol{\Phi}_1> + <\boldsymbol{\Psi}_1,\boldsymbol{\Phi}_2><\boldsymbol{\Lambda}^{m-2}\boldsymbol{\Psi}_1,\boldsymbol{\Phi}_2>,$$
$$m\ge 2, \tag{27}$$

Similarly, for t_1 part, we have an obvious equality $\boldsymbol{V}_{t_1}=[\boldsymbol{V}^{(1)}\ \boldsymbol{V}]$. Furthermore we have $\tilde{\boldsymbol{V}}_{t_1}=[\tilde{\boldsymbol{V}}^{(1)}\ \tilde{\boldsymbol{V}}]$, which means that $\{F_m\}$ ($m\ge -1$) are integrals of motion for system Eq.(25).

On the other hand, it is natural to have that $\tilde{F}_j=\phi_{1j}\psi_{1j}+\phi_{2j}\psi_{2j}$ ($1\le j\le N$) are also integrals of motion for Eq.(22). For ($4N$+1)-dimensional systems Eqs.(22) and (25), we can choose $2N$+1 integrals of motion

$$\tilde{F}_1,\ \tilde{F}_2,\ \cdots,\ \tilde{F}_N,\ F_0,\ F_1,\ \cdots,\ F_N, \tag{28}$$

whose involution is easy to verify. In what follows, we would prove the independence of Eq.(28). Since

$$\begin{cases} \dfrac{\partial \tilde{F}_j}{\partial \phi_{kl}} = \delta_{jl}\psi_{kl}, \quad j,N\ge 1, \quad k=1,2, \\ \left.\dfrac{\partial F_0}{\partial \boldsymbol{\Phi}_1}\right|_{\boldsymbol{\Phi}_1=\boldsymbol{\Phi}_2=\boldsymbol{0},\,\psi_{N+1}=0} = \dfrac{\partial F_m}{\partial \psi_{N+1}} = 0, \quad m\ge 1, \\ \left.\dfrac{\partial F_0}{\partial \boldsymbol{\Phi}_2}\right|_{\boldsymbol{\Phi}_1=\boldsymbol{\Phi}_2=\boldsymbol{0},\,\psi_{N+1}=0} = \boldsymbol{\Psi}_1, \\ \left.\dfrac{\partial F_0}{\partial \psi_{N+1}}\right|_{\boldsymbol{\Phi}_1=\boldsymbol{\Phi}_2=\boldsymbol{0},\,\psi_{N+1}=0} = -1, \\ \left.\dfrac{\partial F_m}{\partial \boldsymbol{\Phi}_1}\right|_{\boldsymbol{\Phi}_1=\boldsymbol{\Phi}_2=\boldsymbol{0},\,\psi_{N+1}=0} = -\boldsymbol{\Lambda}^{m-1}\boldsymbol{\Psi}_2, \ m\ge 1, \\ \left.\dfrac{\partial F_m}{\partial \boldsymbol{\Phi}_2}\right|_{\boldsymbol{\Phi}_1=\boldsymbol{\Phi}_2=\boldsymbol{0},\,\psi_{N+1}=0} = \boldsymbol{\Lambda}^{m}\boldsymbol{\Psi}_1, \ m\ge 1, \end{cases} \tag{29}$$

we have

$$\Delta = \left|\begin{matrix} \dfrac{\partial \tilde{F}_1}{\partial \boldsymbol{\Phi}_1} & \cdots & \dfrac{\partial \tilde{F}_N}{\partial \boldsymbol{\Phi}_1} & \dfrac{\partial F_0}{\partial \boldsymbol{\Phi}_1} & \dfrac{\partial F_1}{\partial \boldsymbol{\Phi}_1} & \cdots & \dfrac{\partial F_N}{\partial \boldsymbol{\Phi}_1} \\ \dfrac{\partial \tilde{F}_1}{\partial \boldsymbol{\Phi}_2} & \cdots & \dfrac{\partial \tilde{F}_N}{\partial \boldsymbol{\Phi}_2} & \dfrac{\partial F_0}{\partial \boldsymbol{\Phi}_2} & \dfrac{\partial F_1}{\partial \boldsymbol{\Phi}_2} & \cdots & \dfrac{\partial F_N}{\partial \boldsymbol{\Phi}_2} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \dfrac{\partial \tilde{F}_1}{\partial \psi_{N+1}} & \cdots & \dfrac{\partial \tilde{F}_N}{\partial \psi_{N+1}} & \dfrac{\partial F_0}{\partial \psi_{N+1}} & \dfrac{\partial F_1}{\partial \psi_{N+1}} & \cdots & \dfrac{\partial F_N}{\partial \psi_{N+1}} \end{matrix}\right|_{\text{sub}}$$
$$= \left|\begin{matrix} \psi_{11} & \cdots & 0 & 0 & -\psi_{21} & \cdots & -\lambda_1^{N-1}\psi_{21} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & \psi_{1N} & 0 & -\psi_{2N} & \cdots & -\lambda_N^{N-1}\psi_{2N} \\ \psi_{21} & \cdots & 0 & 0 & \lambda_1\psi_{11} & \cdots & \lambda_1^{N}\psi_{11} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & \psi_{2N} & 0 & \lambda_N\psi_{1N} & \cdots & \lambda_N^{N}\psi_{1N} \\ 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \end{matrix}\right|$$
$$= (-1)^{N+1}\prod_{i=1}^{2}\prod_{j=1}^{N}\lambda_j\psi_{ij}(\psi_{1j}^2+\psi_{2j}^2)\left|\begin{matrix} 1 & \cdots & \lambda_1^{N-1} \\ \vdots & & \vdots \\ 1 & \cdots & \lambda_N^{N-1}\end{matrix}\right|$$
$$= (-1)^{N+1}\prod_{i=1}^{2}\prod_{j=1}^{N}\lambda_j\psi_{ij}(\psi_{1j}^2+\psi_{2j}^2)\prod_{1\le l<k\le N}(\lambda_k-\lambda_l)$$
$$\ne 0.$$

where ‘sub’ represents $\boldsymbol{\Phi}_1=\boldsymbol{\Phi}_2=\boldsymbol{0},\ \phi_{N+1}=\psi_{N+1}=0$.

Above all, $2N$+1 integrals of motion Eq.(28) are functionally independent. So we have the following theorem:

Theorem 1 The nonlinearized systems Eqs.(22) and (25) are Liouville integrable Hamiltonian systems, whose independent and involutive integrals of motion are given by Eq.(28).

CONCLUSION AND REMARK

In this paper, we present an implicit symmetry constraint Eq.(19) between potential and eigenfunctions for the mKdV equation. After introducing two new independent variables in Eq.(20), the spectral problem Eq.(2) and the adjoint spectral problem Eq.(13) are decomposed into ($4N$+2)-dimensional systems. Furthermore the nonlinearized systems are Hamiltonian systems, and $2N$+1 independent and involutive integrals of motion are given explicitly.

That is to say, the nonlinearized systems are completely integrable in the Liouville sense.

Here only an implicit symmetry constraint of the mKdV equation is considered. However, other implicit symmetry constraints of the mKdV equation can be imposed. We believe that the obtained nonlinearized systems are also Liouville integrable Hamiltonian systems. Moreover we can consider higher-order symmetry constraints of other soliton equations.

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