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NMHV amplitudes in simple electroweak processes

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Abstract: The author applied extended BCF/BCFW rules with fermions to a set of simple electroweak processes in colliders. In such processes, there are two electroweak channels, one with a photon and the other with a Z as the internal particle. Some qualifications are needed. Compact results were obtained for tree-level NMHV amplitudes which include a quark-anti-quark pair and n gluons of the same helicity except one in the final state. In this work, we present a brief review of the BCF/BCFW rules and extensions, and list the NMHV amplitudes.

Key words: NMHV, Electroweak, BCF rules
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INTRODUCTION

Scattering amplitudes in perturbative gauge theories have remarkable properties. At the tree level, maximal helicity violating (MHV) amplitudes in pure QCD can be summarized by the so-called Parke-Taylor formula in one line (Parke and Taylor, 1986; Berends and Giele, 1987; 1988; Mangano *et al*., 1988).

$$A(1^-,2^+,\cdots,i^-,\cdots,n^+)=\frac{\langle 1i\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle (n-1)n\rangle\langle n1\rangle}. \quad (1)$$

Loop corrections are also unexpectedly simple in general (Anastasiou *et al*., 2003; Bern *et al*., 1996; 2003).

It was recently found that one type of topological string theories capture the properties of these amplitudes and that gluon amplitudes are supported on certain algebraic curves when expressed in twistor space (Witten, 2004). Based upon these insights as well as careful inspections of all known field theory results, a new diagrammatic expansion of tree amplitudes in terms of MHV vertices (CSW) was proposed (Cachazo *et al*., 2004) and had been applied to various tree level calculations (Bena *et al*., 2005; Georgiou and Khoze, 2004; Georgiou *et al*., 2004; Khoze, 2004; Kosower, 2005; Wu and Zhu, 2004a; 2004b; Zhu, 2004). This new computation scheme reduces the number of corresponding diagrams as well as computational complexities drastically.

However, real amplitudes may well be much simpler than those suggested by the CSW rules (Bern *et al*., 2004; 2005a; 2005b; Roiban *et al*., 2005). A set of recursion relations was proposed to calculate tree amplitudes (BCF), based upon analysis of one-loop amplitudes and infrared relations (Britto *et al*., 2005a; 2005b; Roiban *et al*., 2005). The BCF rules were soon proved directly by using basic facts of tree diagrams with some help of MHV Feynman diagrams (BCFW) in (Britto *et al*., 2005b). The BCF rules give any tree amplitude as a sum over terms constructed from products of two amplitudes of fewer particles multiplied by a Feynman propagator $i{=}q^2$. In (Luo and Wen, 2005a; 2005b), these recursion relations were extended to include fermions, from which, MHV and MHV amplitudes, and non-MHV amplitudes of processes with six partons are reproduced correctly,

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and may also extend to amplitudes with gravitons (Bedford *et al*., 2005; Cachazo and Svrcek, 2005) and one-loop QCD (Bern *et al*., 2005a; 2005b).

Naturally we would like to apply these rules to electroweak processes, in particular to those with QCD subprocess. In this paper we consider such processes in e^+e^- colliders, with a quark-anti-quark pair and n gluons of the same helicity except one in the final state, the so-called next-MHV (NMHV) amplitudes. In such processes, there are two electroweak channels, one with a photon and the other with a Z as the internal particle. The case for the photon channel can be calculated by a straightforward extension. The BCF rules cannot be applied to the Z channel in general, since Z is a massive particle. However, for the particular cases considered, the rules can be applied with slight modifications. After extracting terms including the mass of the Z particle, the rest of the kinematic terms are similar to those of the photon channel. Relatively simple formulae are then obtained for these amplitudes. In calculating the kinematic amplitudes, we chose two "good" reference lines such that the final results are automatically correct (Luo and Wen, 2005a; 2005b). We found that the results of the NMHV amplitudes only include the sum of two kinds of series. These recursion rules have again served as powerful tools in the analysis of jet physics, and will be useful for detecting new physics in linear collider, particularly in the e^+e^- collider during the next two decades in the future.

REVIEW OF THE BCF/BCFW APPROACH AND EXTENSIONS

Now we give a brief review of the BCF/BCFW approach. Take an n-gluon tree-level amplitude of any helicity configuration. As amplitudes of gluons with all the same helicity vanish, we can always arrange gluons such that the ($n-1$)th gluon has negative helicity and the nth gluon has positive helicity. These two lines will be taken as reference lines. Labeling external particles by i, the following recursion relation was claimed in (Britto *et al*., 2005a):

$$A_n(1,2,\cdots,(n-1)^-,n^+)=\sum_{i=1}^{n-3}\sum_{h=+-}\left[(A_{i+2}(\hat{n},1,2,\cdots,i,-\hat{P}^h_{n,i}) \times\frac{1}{P^2_{n,i}}A_{n-i}(+\hat{P}^{-h}_{n,i},i+1,\cdots,n-2,\widehat{n-1})\right], \quad (2)$$

where

$$\begin{aligned}
P_{n,i}&=p_n+p_1+\cdots+p_i,\\
\hat{P}_{n,i}&=P_{n,i}+\frac{P^2_{n,i}}{\langle n-1|P_{n,i}|n]}\lambda_{n-1}\hat{\lambda}_n,\\
\hat{p}_{n-1}&=p_{n-1}-\frac{P^2_{n,i}}{\langle n-1|P_{n,i}|n]}\lambda_{n-1}\hat{\lambda}_n,\\
\hat{p}_n&=p_n+\frac{P^2_{n,i}}{\langle n-1|P_{n,i}|n]}\lambda_{n-1}\hat{\lambda}_n.
\end{aligned} \quad (3)$$

The formula has a natural meaning in (−−++) signature. Notice that $\hat{P}^2_{n,i}=\hat{p}^2_n=\hat{p}^2_{n-1}=0$, so each tree-level amplitude in Eq.(2) has all external gluons on-shell. Energy-momentum conservation is still preserved. It was shown in (Britto *et al*., 2005b) that reference gluons do not have to be adjacent and that they can be of the same helicity.

To streamline notations, it is expedient to define

$$K^{[r]}_i\equiv p_i+p_{i+1}+\cdots+p_{i+r-1},\ t^{[r]}_i\equiv(K^{[r]}_i)^2$$

And the following relations will be useful for implementing simplifications:

$$\begin{aligned}
\langle A|\hat{P}_{n,i}\rangle&=-\frac{1}{\omega}\langle A|P_{n,i}|n],\\
[\hat{P}_{n,i}|B]&=-\frac{1}{\varpi}\langle n|P_{n,i}|B].
\end{aligned} \quad (4)$$

Factors ω and ϖ always show up in final results in the combination of $\omega\varpi$, which is equal to $\langle n-1|P_{n,i}|n]$. This completes the BCF/BCFW prescription.

These rules can be extended to include fermions, but there are some qualifications. Specifically, one chooses any two external lines, either gluons or fermions, as reference lines; then shifts relevant momenta exactly in the same manner as indicated in Eq.(3) and finally, combines sub-amplitudes of less numbers of external lines together in the same manner as indicated in Eq.(2). But one can take neither two adjacent fermion lines of the same or opposite helicities, or two adjacent gluons of the same helicity, as reference lines. Otherwise one runs into inconsistencies (Luo and Wen, 2005a; 2005b).

THE NMHV AMPLITUDES

Using color decomposition, an n-point amplitude at tree level in pure QCD with colors $a_1,a_2,...,a_n$, external momenta $p_1,p_2,...,p_n$ can be written as

$$M_n = \sum_{\{1,2,\ldots,n\}'} \mathrm{tr}(\boldsymbol{T}^{a_1},\boldsymbol{T}^{a_2},\ldots,\boldsymbol{T}^{a_n}) A(p_1^{\pm},p_2^{\pm},\ldots,p_n^{\pm}) \quad (5)$$

where the sum with the prime is over all $(n-1)!$ non-cyclic permutations of $1,2,...,n$ and the $\boldsymbol{T}$'s are the matrices of the symmetry group in the fundamental representation. The superscript "±" denotes the helicities of external particles. We call "tr" the color factor, and $A(p_1^{\pm},p_2^{\pm},\cdots,p_n^{\pm})$ the kinematic amplitude. The MHV kinematic amplitudes of pure gluons is shown in Eq.(1) and those of two fermions and n gluons can be get from (Georgiou *et al*., 2004).

There are two possible electroweak channels in a simple process as shown in Fig.1, one with a photon as the internal particle, and the other with Z. The amplitude's computation is straightforward when the propagator is a photon. Because the photon is generator of $U(1)$ gauge group and we assume it has $(n-4)$ gluons, the color factor is equal to $(\boldsymbol{T}^{a_1},\boldsymbol{T}^{a_2},\ldots,\boldsymbol{T}^{a_{n-4}})_{i\bar{i}}$ according to Eq.(27) of (Luo and Wen, 2005b). Amplitudes can be obtained from these pure QCD sub-leading amplitudes (Luo and Wen, 2005b). Specifically, the kinematic amplitudes can be written as $A(qeeqg...g)$. Of course, these amplitudes include fermions with two flavors. The corresponding MHV amplitudes can be obtained from (Luo and Wen, 2005b), where all kinds of two or three flavors MHV amplitudes implied there can be easily calculated through the extended BCF rules.

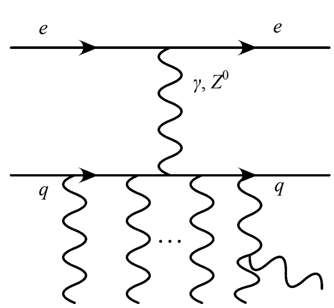

Fig.1 The annihilation of e^+e^- with a quark-anti-quark pair and n gluons in the final state

But Z is a massive particle, the BCFW rules cannot generally be applied to it. However simplifications can be made in our particular case. Because fermions are massless, the Z propagator is effectively

$$\frac{q^2}{q^2-m^2}\times\frac{-ig^{\mu\nu}}{q^2}, \quad (6)$$

here the Z propagator linking to electron lines and quark lines can be factorized out. In this formula the right hand term of the multiplication sign is the Feynman rule of a photon propagator. q^2 is the square of the propagator momentum, and can always be expressed as $\langle ij\rangle[ij]$ for i and j, the order index of e^+ and e^-. Since there is only one Z propagator, the term $q^2/(q^2-m^2)$ can be extracted. The group structure of Z is $U(1)$ effectively, so the remaining parts are identical to that of the photon channel expect for the weak vertex rules. For example, in Eq.(7) $A(q_1^-e_2^+e_3^-q_4^+...g_i^-...)$ has a factor proportional to Qs_w^4/c_w^2 for Z propagator, and to Q for a photon propagator. The weak vertex rules are also different for different helicity.

For calculation of the kinematic amplitudes, we chose two good reference lines. For example, we chose q_1^- and e_2^+ as the reference lines in $A(q_1^-e_2^+e_3^-q_4^+...g_i^-...)$, so that the NMHV amplitudes in sub-diagrams only include two fermions, and we can calculate it simply by BCF rules and extension. But if one chooses two gluons as reference lines there will be too many NMHV diagrams that the calculation is tedious. Considering the proper helicity distribution, we found there are four NMHV kinematic amplitudes as follows (We omit a factor $e^2g_s^{n-6}$, with e the electromagnetic coupling constant and g_s the Yang-Mills coupling):

$$A(q_1^-e_2^+e_3^-q_4^+...g_i^-...) = -\left(\frac{Qs_w^2/c_w^4\langle 23\rangle[23]}{\langle 23\rangle[23]-m_z^2}+Q\right)$$
$$\times\left(\sum_{l=6}^{i}\frac{\langle 13\rangle^2\langle 1i\rangle^3\langle (l-1)l\rangle A_1 s_4^{[l-2]}}{\langle 23\rangle A_2A_3A_4\prod_{j=4}^{n-1}\langle j(j+1)\rangle}\right.$$

$$+\sum_{l=4}^{i-1}\left[\frac{\langle 1i\rangle^3\langle i|K_2^{[l-1]}|2]\langle 3|K_2^{[l-1]}|2]^2}{(\langle 3l\rangle\langle 1|2+3|l]+\langle 13\rangle t_4^{[l-4]})}\right.$$

$$\left.\left.\times\frac{1}{\langle 4|K_2^{[l-1]}|2]\langle 5|K_2^{[l-1]}|2]t_2^{[l-1]}\prod_{j=4}^{n}\langle j(j+1)\rangle}\right]\right),\quad (7)$$

$$A(q_1^- e_2^- e_3^+ q_4^+ ...g_i^- ...)=-Q\left(\frac{(s_w^2-1/2)/c_w^4\langle 23\rangle[23]}{\langle 23\rangle[23]-m_z^2}+1\right)$$

$$\times\left(\sum_{l=6}^{i}\frac{\langle 12\rangle^2\langle 1i\rangle^3\langle (l-1)l\rangle A_1 s_4^{[l-2]}}{\langle 23\rangle A_2A_3A_4\prod_{j=4}^{n-1}\langle j(j+1)\rangle}\right.$$

$$+\sum_{l=4}^{i-1}\left[\frac{\langle 1i\rangle^3\langle i|K_2^{[l-1]}|3]\langle 2|K_2^{[l-1]}|3]^2}{(\langle 2l\rangle\langle 1|2+3|l]+\langle 12\rangle t_4^{[l-4]})}\right.$$

$$\left.\left.\times\frac{1}{\langle 4|K_2^{[l-1]}|3]\langle 5|K_2^{[l-1]}|3]t_2^{[l-1]}\prod_{j=4}^{n}\langle j(j+1)\rangle}\right]\right),(8)$$

$$A(q_1^- ...g_i^- ...q_{n-2}^+ e_{n-1}^- e_n^+)=$$
$$-\left(\frac{Q'(s_w^2-1/2)\langle (n-1)n\rangle[(n-1)n]/c_w^4 s_w^2}{\langle (n-1)n\rangle[(n-1)n]-m_z^2}+Q\right)$$
$$\times\left(\sum_{l=6}^{i}\frac{\langle 1(n-1)\rangle^2\langle l(l-1)\rangle B_1B_2^3}{\langle (n-1)n\rangle B_3B_4B_5B_6}\right.$$
$$\left.+\sum_{l=i}^{n-3}\frac{\langle 1i\rangle^3\langle i|K_{l+1}^{[n-l]}|n]\langle n-1|K_{l+1}^{[n-l]}|n]^2\langle l(l-1)\rangle}{\langle l|K_{l+1}^{[n-l]}|n]\langle l+1|K_{l+1}^{[n-l]}|n]B_7t_{l+1}^{[n-l]}\prod_{j=1}^{n-3}\langle j(j+1)\rangle}\right),\quad (9)$$

$$A(q_1^-\cdots g_i^-\cdots q_{n-2}^+ e_{n-1}^+ e_n^-)=$$
$$-\left(\frac{Q'\langle (n-1)n\rangle[n-1)n]c_w^4}{(\langle (n-1)n\rangle[n-1)n]-m_z^2)}+Q\right)$$
$$\times\left(\sum_{l=2}^{i}\frac{\langle 1n\rangle^2\langle l(l-1)\rangle B_1^3B_2}{\langle (n-1)n\rangle B_3B_4B_5B_6}\right.$$
$$\left.+\sum_{l=i}^{n-3}\frac{\langle 1i\rangle^3\langle i|K_{l+1}^{[n-l]}|n]\langle n|K_{l+1}^{[n-l]}|n-1]^2\langle l(l-1)\rangle}{\langle l|K_{l+1}^{[n-l]}|n]\langle l+1|K_{l+1}^{[n-l]}|n]C_1t_{l+1}^{[n-l]}\prod_{j=1}^{n-3}\langle j(j+1)\rangle}\right),\quad (10)$$

$$A_1=\sum_{m=2}^{l}(\langle im\rangle\langle 1|2+3)m]+\langle i1\rangle\langle 23\rangle[23],$$

$$A_2=\sum_{m=2}^{l}(\langle (l-1)m\rangle\langle 1|2+3)m]+\langle (l-1)1\rangle\langle 23\rangle[23],$$

$$A_3=\sum_{m=2}^{l}(\langle lm\rangle\langle 1|2+3)m]+\langle l1\rangle\langle 23\rangle[23],$$

$$A_4=\sum_{m=2}^{l}\langle 1m\rangle\langle 1|2+3)m],$$

$$s_4^{[l-2]}=(t_2^{[l-2]})^2,$$

$$B_1=\langle i|K_{n-1}^{[l+1]}|n-2]+\frac{\langle (n-1)n\rangle}{\langle 1(n-1)\rangle}\langle i1\rangle[n(n-2)],$$

$$B_2=\langle i(n-2)\rangle+\frac{\langle i|n-1\rangle(t_{n-1}^{[l+1]}+\langle (n-1)n\rangle)}{\langle n-1|K_{n-1}^{[l+1]}|n-2]+\langle (n-1)1\rangle\langle n-1|n|n-2]},$$

$$B_3=\langle l-1|K_{n-1}^{[l+1]}|n-2]+\frac{\langle (n-1)n\rangle}{\langle 1(n-1)\rangle}\langle (l-1)1\rangle[n(n-2)],$$

$$B_4=\langle l|K_{n-1}^{[l+1]}|n-2]+\frac{\langle (n-1)n\rangle}{\langle 1(n-1)\rangle}\langle l1\rangle[n(n-2)],$$

$$B_5=t_{n-1}^{[l+2]}+\frac{\langle (n-1)n\rangle}{\langle 1(n-1)\rangle}\sum_{m=n-1}^{l-1}\langle 1m\rangle[nm],$$

$$B_6=t_{n-2}^{[l+2]}+\frac{\langle (n-1)n\rangle}{\langle 1(n-1)\rangle}\sum_{m=n-2}^{l-1}\langle 1m\rangle[nm],$$

$$B_7=\sum_{m=l}^{n-3}\langle (m+1)(n-1)\rangle\langle 1|K_n^2|m+1]+\langle (n-1)1\rangle t_{l+1}^{[n-l-3]},$$

$$C_1=\sum\langle n(m+1)\rangle\langle 1|K_n^{[2]}|m+1]+\langle 1n\rangle t_{l+1}^{[n-l-3]}.$$

Here m_Z is the mass of Z particle. Q is the charge of quark, so Q=2/3 or −1/3. s_w and c_w are the abbreviation of sinμ_w and cosμ_w, and μ_w is the Weinberg angle. $Q'=\mp 1/2+Qs_w^2$, "−" for up-type quarks and "+" for down-type quarks.

CONCLUSION

We studied a simple electroweak process, where a quark-anti-quark pair and n gluons were produced in e^+e^- annihilation, and calculated the NMHV amplitudes by the BCF approach and extensions. Because of the property of the electroweak interaction, we find the NMHV kinematic amplitudes only include the $A(qeeqg\ldots g)$ and can be expressed by the sum of two kinds of series. When Z is the internal particle, we extract a term involving the mass of Z so that the remainder can be calculated by the BCF rules. It is interesting that we can write out the NMHV amplitudes in a general formula simply. But when you consider the next-to-next MHV amplitudes, there are a great many series so that cannot get a simple result. Another consideration must be involved.

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