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Modified extremal optimization for the hard maximum satisfiability problem*

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Abstract: Based on our recent study on probability distributions for evolution in extremal optimization (EO), we propose a modified framework called EOSAT to approximate ground states of the hard maximum satisfiability (MAXSAT) problem, a generalized version of the satisfiability (SAT) problem. The basic idea behind EOSAT is to generalize the evolutionary probability distribution in the Bose-Einstein-EO (BE-EO) algorithm, competing with other popular algorithms such as simulated annealing and WALKSAT. Experimental results on the hard MAXSAT instances from SATLIB show that the modified algorithms are superior to the original BE-EO algorithm.

Key words: Extremal optimization (EO), Evolution, Probability distributions, Maximum satisfiability (MAXSAT) problem
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1 Introduction

It is well known that thousands of real-world computational problems, ranging from computer chip verification, planning, and scheduling to protein folding, fiber optics routing, etc., can be formulated as satisfiability (SAT) or maximum satisfiability (MAXSAT) problems (Gomes and Selman, 2002). The SAT problem is the first one proved to show the NP-complete property (Garey and Johnson, 1979). During the past few decades, the SAT problem and its modified versions have been challenging to both computer science and statistical physics societies (Biroli *et al.*, 2002; Hartmann and Weigt, 2005; Selman, 2008; Altarelli *et al.*, 2009). Based on the concepts and techniques of statistical physics, computer scientists and physicists have made great efforts in the following two directions.

The first one is analyzing the typical-case behavior of SAT and MAXSAT problems (Cheeseman *et al.*, 1991; Monasson *et al.*, 1999; Zhang, 2001; Mézard *et al.*, 2002). In fact, differing from the traditional worst-case analysis of computational complexity that NP-complete problems require exponential time to solve in the worst case, the typical-case behavior (Monasson *et al.*, 1999) is characterized as a phase transition pattern. More specifically, the computational cost of the heuristic search algorithms for solving the random three-satisfiability (3-SAT) problem is characterized as the ‘easy-hard-easy’ phase transition, which indicates that the highest computational complexity corresponds to the region near some critical values of a parameter $\alpha=m/n$ (number of clauses/number of variables). The experimental and analytic results (Mézard *et al.*, 2002) have shown that the instance of the critical value $\alpha_c\approx4.267$ for 3-SAT and the random instances close to α_c are the most difficult to solve. Zhang (2001) has studied the relationship between the phase transition of 3-SAT and

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that of MAX 3-SAT.

The second is how to understand and develop effective and efficient algorithms for the SAT problem and its corresponding optimization problem, i.e., the MAXSAT problem (Kirkpatrick *et al*., 1983; Hansen and Jaumard, 1990; Selman and Kautz, 1993; Selman *et al*., 1994; Barthel *et al*., 2003; Semerjian and Monasson, 2003; Zhang, 2004; Seitz *et al*., 2005; Alava *et al*., 2008). In fact, the SAT and MAXSAT problems have been widely used as a testbed for new algorithms. The popular physics-inspired stochastic local search algorithms for the SAT and MAXSAT problems include simulated annealing (SA) (Hansen and Jaumard, 1990), focused metropolis search (FMS) (Seitz *et al*., 2005), WALKSAT (Selman *et al*., 1994), survey propagation (SP) (Mézard *et al*., 2002), etc. This paper follows the second direction, focusing on the improvement of some reported competitive algorithms for solving the MAXSAT problem.

Originally inspired by far-from-equilibrium dynamics of self-organized criticality (SOC) (Bak *et al*., 1987; Bak and Sneppen, 1993), a recently developed method called extremal optimization (EO) (Boettcher and Percus, 2000; 2001a) provides a novel insight into the optimization domain. This method merely selects against the bad, instead of favoring the good, either randomly or according to a power-law distribution. The basic EO algorithm and its modified versions (Middleton, 2004; Zhou *et al*., 2005; Hamacher, 2007) have been successfully applied to graph partitioning (Boettcher and Percus, 2001b), graph coloring (Boettcher and Percus, 2004), spin glasses (Middleton, 2004; Hamacher, 2007), Lennard-Jones clusters (Zhou *et al*., 2005), etc. The survey on EO is referred to our recent work (Zeng and Lu, 2009). Recently, a modified EO algorithm called Bose-Einstein-EO (BE-EO) (Menaï and Batouche, 2006) has been proposed to solve the MAXSAT problem. The basic idea behind BE-EO is to sample the initial configurations set based on BE distribution to the original τ-EO search process. Experimental results on both random and structured MAXSAT instances demonstrate BE-EO's superiority to more elaborate stochastic optimization methods such as SA (Hansen and Jaumard, 1990), greedy SAT (GSAT) (Selman and Kautz, 1993), WALKSAT (Selman *et al*., 1994), and tabu search (Glover, 1989).

To the best of our knowledge, most of the existing EO algorithms and the modified versions apply power-law distributions as their evolutionary rules. However, it has not been proved from the theoretical perspective. In our recent publication (Zeng *et al*., 2010), we have studied the probability distributions for evolution in a modified EO for the travelling salesman problem (TSP). The experimental results on uniform and non-uniform TSP instances have shown that the power-law distribution is not the only good evolutionary probability distribution, while others, such as exponential and hybrid distributions, are better or at least competitive ones. In fact, this preliminary result is consistent with the observation that, for the original EO algorithm, the power-law distribution may not be optimal (Heilmann *et al*., 2004; Hoffmann *et al*., 2004). To further demonstrate the reasonableness of this observation, we extend the basic idea behind the previous study (Zeng *et al*., 2010) to the hard MAXSAT problem. More specifically, we propose a modified EO framework, called EOSAT, to solve the hard MAXSAT problem by generalizing the evolutionary probability distribution in BE-EO. The superiority of the modified algorithms to the reported BE-EO algorithm is demonstrated by the experimental results on the hard MAXSAT instances from SATLIB.

2 The MAXSAT problem and probability distributions

2.1 MAXSAT problem

Random *K*-SAT instances (Biere *et al*., 2009) are constructed by selecting independently and uniformly at random *m* clauses over the *n* Boolean variables, where each clause consists of *K* literals, i.e., Boolean variables or their negations. The parameter controlling the satisfiability of an instance is $\alpha=m/n$, which is the ratio of the number of clauses to the number of variables. When $K=3$, these instances are called 3-SAT instances. Let $X=\{x_1, x_2, \ldots, x_n\}$ be a set of Boolean variables, where x_i can take the value 0 (false) or the value 1 (true). Let $C=\{C_1, C_2, \ldots, C_m\}$ be a set of clauses, each of which is a disjunction of literals l_{ij}. A formula *F* is called conjunctive normal form if it is a conjunction of the clauses:

$$F = \bigwedge_{i=1}^{m} \left(\bigvee_{j=1}^{|C_i|} l_{ij} \right), \tag{1}$$

where $K=|C_i|$ is the number of literals in a clause C_i, and l_{ij} is a Boolean variable x_i or its negation $\overline{x}_i$. Clearly, a clause is satisfied if at least one of its literals is true and F is satisfied if all the clauses are satisfied. A weighted formula is a pair WF={F, $\boldsymbol{W}$}, where $\boldsymbol{W}=(w_i)\in\mathbb{N}^m$ is an integer vector and w_i is the weight of the clause C_i. When $w_i=1$ for each clause, the formula is called an unweighted formula.

K-SAT is a decision problem, that is, to find whether there exists an assignment that satisfies all the clauses in F, while maximum K-satisfiability (MAX-K-SAT) is the optimization counterpart of K-SAT, i.e., to find an assignment S to maximize the number of satisfied clauses (Zhang, 2001). It is obvious that MAX-K-SAT is more general than K-SAT, because its solution can be used to answer the question of the K-SAT problem, but not vice versa (Zhang, 2004). It has been shown that the MAX-K-SAT is a typical NP-hard problem when $K\geq 2$ (Cheeseman *et al.*, 1991). As an extended version of MAX-K-SAT, the weighted MAX-K-SAT problem can be defined to find an assignment S to maximize the total weight of the satisfied clauses, i.e., to minimize W_u, the total weight of unsatisfied clauses:

$$W_{\mathrm{u}}(C,\boldsymbol{W},S)=\min\sum_{C_j(S)=0} w_j. \tag{2}$$

2.2 Probability distributions

Originating from quantum physics, the BE distribution (Haken and Wolf, 1996) describes the statistical behavior of bosons (integer spin particles). At low temperatures, bosons can behave very differently than fermions because an unlimited number of them can collect into the same energy state, a phenomenon called 'condensation'. In the context of combinatorial description (Szedmak, 2001), this distribution can be defined as

$$p_x=\frac{1}{(n+1)\begin{pmatrix} n \\ x[V]\end{pmatrix}},\quad \forall x\in\{0,1\}^n, \tag{3}$$

where p_x is the probability distribution of x in the space $\{0, 1\}^n$, $V=\{1, 2, \ldots, n\}$ is a base set for a given n, $x=\{x_1, x_2, \ldots, x_n\}$ is a set of Boolean variables, and $x[V]=\sum_{i\in V}x_i$ is the number of the variables of x equal to 1 in V. Its conditional probability is

$$p(x_j=1)=\frac{x[S]+1}{(j-1)+2},\quad S=\{1,2,\ldots,j-1\}. \tag{4}$$

Here, we introduce the probability distributions used for evolution in EO (Zeng *et al.*, 2010). The power-law $P_{\mathrm{p}}(k)$, exponential distribution $P_{\mathrm{e}}(k)$, and hybrid distribution $P_{\mathrm{h}}(k)$ are described as follows:

$$P_{\mathrm{p}}(k)=k^{-\tau},\quad 1\leq k\leq n, \tag{5}$$

$$P_{\mathrm{e}}(k)=\mathrm{e}^{-\mu k},\quad 1\leq k\leq n, \tag{6}$$

$$P_{\mathrm{h}}(k)=\mathrm{e}^{-hk}k^{-h},\quad 1\leq k\leq n, \tag{7}$$

where τ, μ, and h are all positive constants for a specific problem with size n.

3 A modified EO framework for the MAXSAT problem

It has been proved that for the MAXSAT problem, only the BE distribution can guarantee that an initial assignment set is generated with an arbitrary proportion of 1's and 0's (Szedmak, 2001). Moreover, the experimental results (Menaï and Batouche, 2006) on random and structured MAXSAT instances have shown that the BE-EO algorithm, starting from BE-based initial configurations, can provide better performance than τ-EO from uniformly random ones. Therefore, a BE-based assignment will be used as the initial configuration of the proposed framework in this study. It is constructed by the following procedure called BEICG (BE-based initial configuration generator).

Algorithm 1 Bose-Einstein-based initial configuration generator

Input: a set of n Boolean variables x_i
Output: a random BE-based assignment of the n Boolean variables

```
u=0 // number of 1's
for i=1 to n
    p_i = (u+1)/[(i-1)+2]
    Generate randomly a real number a in [0, 1]
    if p_i>a
        x_i=1, u=u+1
    else x_i=0
    end if
end for
```

According to Boettcher and Percus (2000; 2001a), EO is applicable to these problems whose cost can be decomposed into contributions from individual degrees of freedom. In other words, the objective or energy function of the problems under consideration should be additive over the local fitness of each degree of freedom. For a given configuration S of the SAT problem, the local fitness λ_i of each variable x_i is defined as follows:

$$\lambda_i = -\frac{\sum_{x_i \in C_j,\, C_j(S)=0} w_j}{\sum_{x_i \in C_k} w_k}. \quad (8)$$

In other words, the local fitness is defined as the ratio of the sum of weights of unsatisfied clauses in which the variable x_i appears to the sum of weights of clauses connected to this variable.

The global fitness $C(S)$ is defined as the sum of the contribution from each variable, i.e.,

$$C(S) = -\sum_{i=1}^{n}\left(\lambda_i \sum_{x_i \in C_k} w_k\right) = -\sum_{i=1}^{n}(c_i \lambda_i), \quad (9)$$

where $c_i = \sum_{x_i \in C_k} w_k$ is a constant for a given problem. Clearly, $C(S)$ is a linear combination of local fitness λ_i, which is consistent with the observation concerning the fitness definition (Boettcher and Percus, 2001a).

Inspired by our recent study (Zeng *et al.*, 2010) on probability distributions for evolution in modified EO for TSP, we propose a modified EO framework, called EOSAT, for the MAXSAT problem, by generalizing the evolutionary probability distribution in BE-EO. The details of EOSAT are described below:

1. Generate a BE-based initial configuration S by BEICG and set $S_{\text{best}}=S$, $C(S_{\text{best}})=C(S)$.

2. If S satisfies all clauses, then $S_{\text{best}}=S$ and go to step 6.

3. For current configuration S,

(1) Evaluate λ_i for each variable x_i and rank all variables according to λ_i, i.e., find a permutation Π_1 of the labels i such that $\lambda_{\Pi_1(1)} \geq \lambda_{\Pi_1(2)} \geq \ldots \geq \lambda_{\Pi_1(n)}$;

(2) Select a rank $\Pi(k)$ according to a probability distribution $P(k)$, $1 \leq k \leq n$, and denote the corresponding variable as x_j;

(3) Flip the value of x_j and set $S_{\text{new}}=S$ in which the value of x_j is flipped;

(4) If $C(S_{\text{new}}) \leq C(S_{\text{best}})$, then $S_{\text{best}}=S_{\text{new}}$, $C(S_{\text{best}})=C(S_{\text{new}})$;

(5) Accept $S=S_{\text{new}}$ unconditionally.

4. Repeat step 3 until the preset maximal number of iterations, N.

5. Store S_{best} and $C(S_{\text{best}})$, and obtain $C(S_{\text{B}})=\min\{C(S_{\text{best}})\}$, where S_{B} is the best solution found so far.

6. Repeat steps 1–5 until the preset maximal sample size, R.

7. Return S_{B} and $C(S_{\text{B}})$.

It is clear that $P(k)$ plays a critical role in controlling the performance of the above framework. The specific algorithms that depend on the evolutionary probability distribution are defined in Table 1. In particular, the BE-EO algorithm (Menaï and Batouche, 2006) is the special case of the above framework. In other words, the proposed framework can be viewed as a generalized version of BE-EO. Furthermore, on the basis of the previous study on TSP, this work will further demonstrate that other probability distributions, rather than power law, may be competitive or even better choices used in EO-similar methods for hard SAT and MAXSAT problems. To some extent, this work is an extension of the basic idea behind the previous study (Zeng *et al.*, 2010) concerning the hard MAXSAT problem. Additionally, as a generalized version of BE-EO, the proposed framework EOSAT provides a novel stochastic local search method for the hard MAXSAT problem. Its effectiveness will be demonstrated by the experimental results in the next section.

Table 1 The modified EO algorithms with different evolutionary probability distributions for the MAXSAT problem

Algorithm	$P(k)$
BE-EO	$k^{-\tau}$
BE-EEO	$e^{-\mu k}$
BE-HEO	$e^{-hk}k^{-h}$

The control parameters, such as τ, μ, and h used in BE-EO, BE-EEO, and BE-HEO, play an analogous role to the proportion p of random and greedy moves in WALKSAT (Selman *et al.*, 1994), and the noise parameter η in FMS (Seitz *et al.*, 2005). Specifically, for large values of τ, BE-EO becomes entropic, while for small values, it is greedy. There are similar effects of μ and h on BE-EEO and BE-HEO, respectively. In the next section, we will discuss the empirical determination of the optimal values of the control parameters for each modified EO algorithm.

4 Experimental results and analysis

To demonstrate the effectiveness of the EOSAT, we choose the hard MAXSAT problem instances from SATLIB (Hoos and Stuzle, 1999) as a testbed. These unsatisfiable instances are represented as uuf-n:m here, in which n is the number of the variables and m is the number of the clauses. For each instance, the optimal number of unsatisfied clauses is 1. Therefore, we focus on the optimization problem, MAX-3-SAT, to find an assignment to maximize the number of satisfied clauses. In other words, MAX-3-SAT is equivalent to minimizing the number of unsatisfied clauses. Here, 10 instances of each problem are chosen for testing, and each instance goes through 10 independent runs. In all modified EO algorithms, we set R=50 and N=1000. We implemented all algorithms in MATLAB 7.6 on a Pentium 1.86 GHz PC with a dual-core processor T2390 and 2 GB RAM running the Windows Vista Basic system. The performance of these algorithms is measured by the best, mean, and worst errors denoted as e_b, e_m, and e_w, respectively, and the detailed results are shown in Table 2. The errors are defined as e_b (%)=100×(m_b−m_o)/m, e_m (%)=100×(m_m−m_o)/m, and e_w (%)=100×(m_w−m_o)/m, where m_b, m_m, and m_w are the minimal, mean, and maximal numbers of unsatisfied clauses over 10 independent runs, respectively, and m_o is the optimal solution.

The experimental results in Menaï and Batouche (2006) on random and structured MAXSAT instances have shown that BE-EO can provide better or at least competitive performance compared with more elaborate stochastic optimization methods, such as SA (Hansen and Jaumard, 1990), GSAT (Selman and Kautz, 1993), WALKSAT (Selman *et al.*, 1994), and tabu search (Glover, 1989). This study focuses on comparing the modified EO algorithms including BE-EEO and BE-HEO with BE-EO (Menaï and Batouche, 2006), a reported successful algorithm by experiments on hard MAXSAT instances with α=m/n ranging from 4.260 to 4.360 near the critical threshold of phase transition α_c≈4.267. Table 2 shows that BE-EEO and BE-HEO provide better performance than BE-EO with the same CPU time. This indicates that besides the power law used in BE-EO, the exponential and hybrid distributions are also possible good choices for evolution in the EOSAT framework. More interestingly and importantly, these distributions appear to be more appropriate than the power law used as an evolutionary mechanism in the proposed framework.

Table 2 Comparison of the modified EO algorithms for the hard MAXSAT instances near the critical threshold of phase transition α_c≈4.267

Problem	α	Algorithm	e_b (%)	e_m (%)	e_w (%)
uuf-50:218	4.360	BE-EO	1.38	2.38	3.21
		BE-EEO	0.46	1.88	2.75
		BE-HEO	0.00	1.93	2.75
uuf-75:325	4.333	BE-EO	1.85	2.65	3.37
		BE-EEO	1.23	1.94	2.46
		BE-HEO	1.23	1.97	2.46
uuf-100:430	4.300	BE-EO	1.86	2.63	2.80
		BE-EEO	1.16	1.88	2.56
		BE-HEO	0.46	1.93	2.56
uuf-125:538	4.304	BE-EO	1.86	2.70	3.35
		BE-EEO	1.30	2.08	3.16
		BE-HEO	1.30	1.86	2.23
uuf-150:645	4.300	BE-EO	2.17	2.71	3.41
		BE-EEO	1.40	2.05	2.64
		BE-HEO	0.78	1.89	2.64
uuf-175:753	4.303	BE-EO	2.92	3.33	3.98
		BE-EEO	1.73	2.30	2.67
		BE-HEO	1.59	2.31	2.52
uuf-200:860	4.300	BE-EO	3.49	3.85	4.30
		BE-EEO	2.44	2.72	3.14
		BE-HEO	2.33	2.87	3.26
uuf-225:960	4.267	BE-EO	2.81	3.48	4.17
		BE-EEO	2.19	2.64	3.44
		BE-HEO	1.88	2.40	3.23
uuf-250:1065	4.260	BE-EO	3.09	3.51	4.38
		BE-EEO	1.78	2.28	2.72
		BE-HEO	2.06	2.44	2.90

e_b, e_m, and e_w are the best, mean, and worst errors, respectively

The search dynamics of the modified EO algorithms defined in Table 1 when R=1 and N=10^4 are shown in Fig. 1. All the algorithms start from the same initial configuration for the uuf-50:218 instance. From these optimization dynamics, we find that they descend sufficiently fast to the metastable states first, and then proceed to explore the complex landscapes with different fluctuations to approach lower metastable states even ground states. In fact, these fluctuations depend on the evolutionary probability distributions adopted by the corresponding algorithms. Different from other stochastic local algorithms, e.g., SA, the EO-similar algorithms always select these bad variables for updating, in order to construct a new configuration and accept the new configuration unconditionally. These evident characteristics lead to the non-equilibrium feature of EO-similar algorithms. In addition, we are also interested in the typical dynamical processes of $C(S_B)$ as the sample size increases in these algorithms. For example, Fig. 2 shows these processes for the same uuf-50:218 instance. Compared to BE-EO, the modified algorithms, including BE-EEO and BE-HEO, can reach lower states.

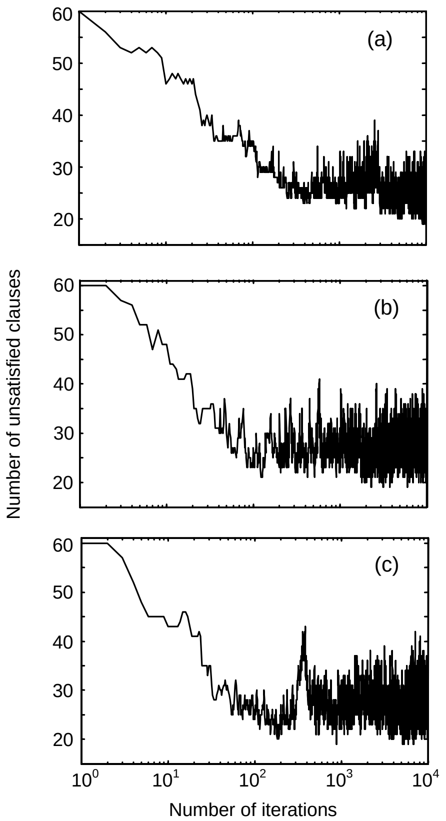

Fig. 1 Search dynamics of BE-EO (a), BE-EEO (b), and BE-HEO (c), starting from the same initial configuration when the preset maximal sample size R=1 and the preset maximal number of iterations N=10^4 for the same uuf-50:218 instance

Fig. 3 shows the resulting performance of BE-EO, BE-EEO, and BE-HEO on the uuf-50:218 instance by 100 independent runs. In each algorithm, we set R=50 and N=1000. The histograms present the frequencies with which a particular number of unsatisfied clauses are obtained over the trial runs. Obviously, the frequencies of optimal and near-optimal configurations obtained by BE-EEO and BE-HEO are higher than that obtained by BE-EO. In particular, BE-HEO obtains the optimal configurations while BE-EEO obtains near-optimal solutions more often than BE-EO. Fig. 3d shows that BE-EEO and BE-HEO have more probabilities to approach lower metastable states even ground states than BE-EO. In this sense, the algorithms including BE-EEO and BE-HEO are superior to the original BE-EO.

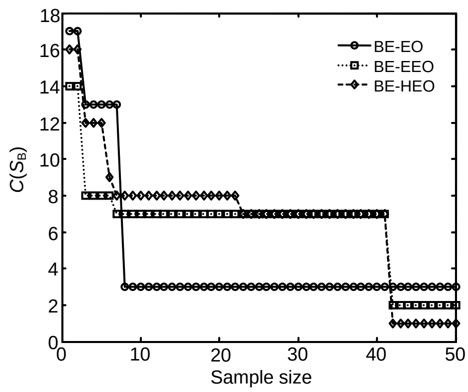

Fig. 2 Typical dynamical processes of $C(S_B)$ in BE-EO, BE-EEO, and BE-HEO when the preset maximal sample size R=50 and the preset maximal number of iterations N=10^3 for the same uuf-50:218 instance

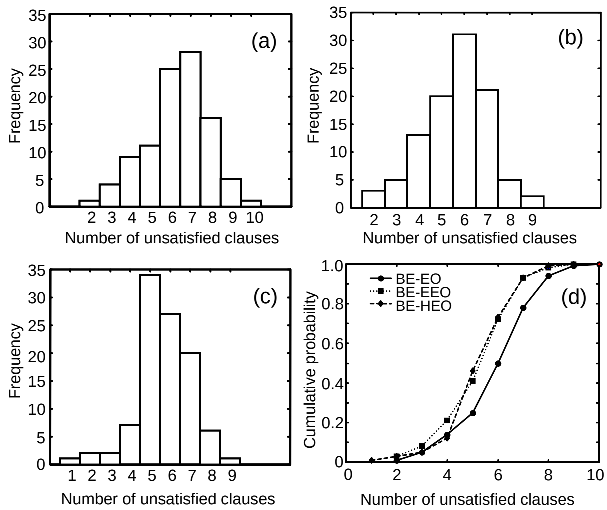

Fig. 3 Comparison of 100 trial runs using BE-EO (a), BE-EEO (b), and BE-HEO (c) on the uuf-50:218 instance and the corresponding cumulative probability (d)

The histograms present the frequency with which a particular number of unsatisfied clauses are obtained over the trial runs

As analyzed in Section 3, the performance of the modified EO algorithms under the proposed EOSAT framework depends on the parameters τ, μ, and h controlling the probability distribution $P(k)$. As a consequence, how to determine the optimal value of the control parameter used in each algorithm is an important issue. In the work of Menaï and Batouche (2006), the optimal value of τ in BE-EO has been studied numerically. Here, a similar method is applied to determine the optimal values of μ and h in BE-EEO and BE-HEO, respectively. Fig. 4 illustrates the effects of μ and h on the performances of BE-EEO and BE-HEO for the uuf-50:218 instance. The performances are evaluated by the worst, average, and best errors over 10 independent runs, varying μ and h between 0.06 and 0.60. For n=50, the optimal value of μ in BE-EEO is on average from 0.46 to 0.50, while for other problem instances (n=75–250), it is between 0.14 and 0.42. Similarly, the optimal value of h in BE-HEO is on average from 0.475 to 0.485 for n=50, while for others it is between 0.13 and 0.46. Note that the optimal values of the control parameters for average performances may be a bit different from those for best performances. For example, for BE-HEO, the best solution can be obtained when h=0.34, which is a bit different from the optimal values 0.475–0.485 for average performance.

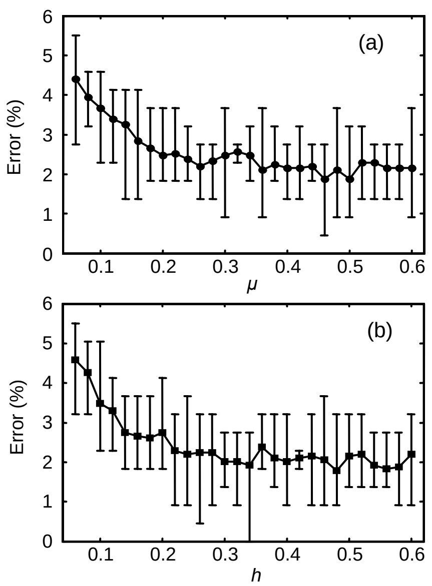

Fig. 4 The control parameters μ in BE-EEO (a) and h in BE-HEO (b) vs. the corresponding performance, which is evaluated by the worst, average, and best errors over 10 independent runs for the same uuf-50:218 instance

5 Conclusions

We present a modified EO framework called EOSAT to solve the hard MAXSAT problem, a generalized version of the SAT problem. On the basis of our recent study (Zeng *et al.*, 2010) on the probability distributions in modified EO for TSP, the EOSAT generalizes the evolutionary probability distribution in BE-EO, an effective stochastic local algorithm for the MAXSAT problem. Experimental results on random MAXSAT instances near the phase transition have shown that the proposed algorithms, including BE-EEO and BE-HEO under the EOSAT framework, outperform BE-EO. This further demonstrates that other probability distributions, rather than power law, may be the competitive or even better evolutionary probability distributions used in EO-similar methods for the hard SAT and MAXSAT problems. Furthermore, as a generalized version of BE-EO (Menaï and Batouche, 2006), EOSAT provides another effective stochastic local search method for the hard MAXSAT problem.

There are some open issues, however, for future work. The first is to further test the effectiveness of the proposed framework on some large-size and hard MAXSAT problem instances, and to further improve its performance by designing a more effective evolutionary mechanism. Understanding more exactly why other evolutionary probability distributions are more effective than the power law is another issue. In addition, the scaling of the EO algorithms and modified versions to the MAXSAT problem will be investigated in future work.

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