



Uniform modeling of parameter dependent nonlinear systems

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Abstract: This paper addresses the problem of approximating parameter dependent nonlinear systems in a unified framework. This modeling has been presented for the first time in the form of parameter dependent piecewise affine systems. In this model, the matrices and vectors defining piecewise affine systems are affine functions of parameters. Modeling of the system is done based on distinct spaces of state and parameter, and the operating regions are partitioned into the sections that we call ‘multiplied simplices’. It is proven that this method of partitioning leads to less complexity of the approximated model compared with the few existing methods for modeling of parameter dependent nonlinear systems. It is also proven that the approximation is continuous for continuous functions and can be arbitrarily close to the original one. Next, the approximation error is calculated for a special class of parameter dependent nonlinear systems. For this class of systems, by solving an optimization problem, the operating regions can be partitioned into the minimum number of hyper-rectangles such that the modeling error does not exceed a specified value. This modeling method can be the first step towards analyzing the parameter dependent nonlinear systems with a uniform method.

Key words: Parameter dependent nonlinear systems, Approximation method, Parameter dependent piecewise affine systems, Modeling

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1 Introduction

Recently, there has been growing interest in the study of hybrid systems; it is due to the fact that hybrid systems provide a unified framework for describing processes involving continuous dynamics and discrete events. Therefore, this class of systems is sufficiently expressive in modeling a wide range of real world plants. Piecewise affine (PWA) systems form a special class of hybrid systems in which the continuous dynamics within each discrete mode is affine and mode switching always occurs at specified subsets of the state space. The primary motivation for studying PWA systems comes from the fact that a large class of nonlinear systems, appearing frequently in engineering applications, can be approximated by PWA systems. There are two methods for representing a PWA system: conventional (Chien and Kuh, 1977) and canonical (Julian *et al.*, 2000; Storace and

de Feo, 2005a; 2005b; Bergami *et al.*, 2006; Storace and Bizzarri, 2007). PWA approximation in canonical form is a fundamental step in PWA synthesis of a nonlinear function as an integrated circuit (di Federico *et al.*, 2010; Poggi, 2010); approximate syntheses of cellular nonlinear networks (Storace *et al.*, 2003) and nonlinear multiport resistors (Storace *et al.*, 2002) are simple examples of canonical PWA approximation. PWA maps have universal approximation properties (Lin and Unbehauen, 1992; Breiman, 1993). In addition, PWA systems are equivalent to several classes of hybrid systems (Heemels *et al.*, 2001; Bemporad, 2004). This property allows using all the analysis and synthesis tools developed for PWA systems for all the other equivalent subclasses of hybrid systems. Furthermore, PWA systems allow using tractable mathematical tools for analysis and synthesis; thus, piecewise affine systems are very powerful tools in the modeling and analysis of nonlinear systems.

Parameter dependent piecewise affine (PD-

PWA) systems are a generalization of PWA systems, in which their state matrices are functions of parameters. Since the class of PWA systems has been known as a powerful framework for approximating nonlinear systems, the class of PD-PWA systems can be considered as an appropriate framework for modeling of parameter dependent nonlinear (PD-NL) systems. Although in the last decade analysis and synthesis of PD-PWA systems has attracted considerable attention (Lin and Antsaklis, 2003a; 2003b; Zhai *et al.*, 2003; Gao and Chen, 2009; Thomas *et al.*, 2009; Zhang *et al.*, 2009), to our best knowledge, no significant work is available on PD-PWA modeling of a given PD-NL system. The available methods are just weak extensions of PWA modeling of nonlinear systems. As an example, by expanding the state vector into a new vector which includes states and parameters, the problem of approximation of PD-NL systems has been addressed (Storace and de Feo, 2005a; 2005b; Storace and Bizzarri, 2007). It is known that when the number of states increases, the complexity of PWA systems grows rapidly, so the approximation method presented by Storace and de Feo (2005a; 2005b) and Storace and Bizzarri (2007) generates many subsystems for describing the approximated system, which leads to heavy computational burdens.

In this paper, by defining multiplied simplex, a new modeling method is presented for approximating PD-NL systems in the form of PD-PWA systems, in which the state matrices are affine functions of parameters. In this approach, the spaces of state and parameter are considered separately. Separating spaces of state and parameter and partitioning operating regions into multiplied simplices cause the number of subsystems that describe the approximated system to decrease with respect to the approximated system presented by Storace and de Feo (2005a; 2005b) and Storace and Bizzarri (2007). It is proven that the approximated model is continuous on the boundaries of simplices in the state space and parameter space. In addition, the introduced PD-PWA functions can approximate continuous functions on a compact partitioned domain by a continuous function with arbitrary accuracy. It is shown that for a Lipschitz continuous function, the upper bound of the modeling error can be calculated based on the Lipschitz constant of the function. For this class of functions, by solving an optimization problem, the

operating region of the system can be partitioned into the minimum number of simplices such that the modeling error achieved is less than a pre-defined value. After that, it is shown through some examples that for a more refined partition and consequently a smaller modeling error, the behavior of the approximated system is more similar to the original one from both quantitative and qualitative points of view.

2 Notations and preliminaries

Consider the hyper-rectangle

$$Y = \{\mathbf{y} \in \mathbb{R}^n \mid a_i \leq y_i \leq b_i, i = 1, 2, \dots, n\}.$$

Let $\delta_i = b_i - a_i$ and vector $\mathbf{V}_0 = [a_1, a_2, \dots, a_n]^T$ be the corner of hyper-rectangle Y .

Definition 1 Let $\mathbf{y}^0, \mathbf{y}^1, \dots, \mathbf{y}^n$ be $n+1$ points in the n -dimensional space. A simplex $\Delta(\mathbf{y}^0, \mathbf{y}^1, \dots, \mathbf{y}^n)$ is defined as

$$\Delta(\mathbf{y}^0, \mathbf{y}^1, \dots, \mathbf{y}^n) = \left\{ \mathbf{y} : \mathbf{y} = \sum_{i=0}^n \lambda_i \mathbf{y}^i \right\}, \quad (1)$$

where $0 \leq \lambda_i \leq 1, i \in \{0, 1, \dots, n\}, \sum_{i=0}^n \lambda_i = 1$.

In this paper, we refer only to a proper simplex that cannot be contained in an $(n-1)$ -dimensional hyper-plane (Julian *et al.*, 1999).

Now, consider S as a typical simplex in Y whose vertices are specified as follows:

$$\mathbf{V}_k = \mathbf{V}_0 + \mathbf{U}_k, k = 0, 1, \dots, n, \mathbf{U}_k = \mathbf{D} \sum_{i=0}^k \mathbf{e}_{r_i}, \quad (2)$$

where $\mathbf{D} = \text{diag}\{\delta_1, \delta_2, \dots, \delta_n\}, r_i \in \{1, 2, \dots, n\} (r_i \neq r_j \text{ if } i \neq j), \mathbf{e}_{r_i}$ is the r_i th unitary vector and $\mathbf{e}_{r_0} = \mathbf{0}$. As there are $n!$ different ways to choose the n -tuples (r_1, r_2, \dots, r_n) , each hyper-rectangle is in turn subdivided into $n!$ non-overlapping simplices (Julian *et al.*, 1999).

Having vertices $\mathbf{V}_k (k=0, 1, \dots, n)$, the simplex S can be written as $S = \Delta(\mathbf{V}_0, \mathbf{V}_1, \dots, \mathbf{V}_n)$ and based on Eq. (1),

$$\forall \mathbf{y} \in S, \mathbf{y} = \sum_{k=0}^n \lambda_k (\mathbf{V}_0 + \mathbf{U}_k) = \mathbf{V}_0 + \sum_{k=0}^n \lambda_k \mathbf{U}_k.$$

This implies

$$\sum_{k=0}^n \lambda_k \mathbf{U}_k = \mathbf{y} - \mathbf{V}_0, \quad \sum_{k=0}^n \lambda_k = 1,$$

or

$$\begin{bmatrix} \mathbf{U}_0 & \dots & \mathbf{U}_n \\ 1 & \dots & 1 \end{bmatrix} \begin{bmatrix} \lambda_0 \\ \vdots \\ \lambda_n \end{bmatrix} = \begin{bmatrix} \mathbf{y} - \mathbf{V}_0 \\ 1 \end{bmatrix}. \quad (3)$$

Since $\mathbf{U}_1, \mathbf{U}_2, \dots, \mathbf{U}_n$ are linear independent vectors belonging to \mathbb{R}^n , then Eq. (3) has a unique solution and we can find λ_k as an affine function of $\mathbf{y} - \mathbf{V}_0$, or

$$\lambda_k = \mathbf{q}_k (\mathbf{y} - \mathbf{V}_0) + s_k, \quad k = 0, 1, \dots, n, \quad (4)$$

where $\mathbf{q}_k^T \in \mathbb{R}^n$ and $s_k \in \mathbb{R}$. For simplices with similar \mathbf{U}_k , the factors \mathbf{q}_k and s_k can be calculated uniquely.

3 Problem formulation

Consider the following parameter dependent nonlinear system:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{p}), \quad (5)$$

where $\mathbf{x} \in X \subset \mathbb{R}^n$ and $\mathbf{p} \in P \subset \mathbb{R}^m$ are state and parameter vectors respectively, X and P are polytopic regions called ‘operating regions’ and $\mathbf{f}(\mathbf{x}, \mathbf{p}): \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$. In the first step, for systematic modeling, X and P are subdivided into equal hyper-rectangles separately. Then again, each resulting hyper-rectangle, in both state and parameter spaces, is subdivided into simplices via specifying their vertices using Eq. (2). Now, based on this partitioning, the PD-PWA approximation of $\mathbf{f}(\mathbf{x}, \mathbf{p})$ is presented as follows:

$$\begin{aligned} \mathbf{f}_{\text{PWL}}(\mathbf{x}, \mathbf{p}) &= \mathbf{f}_{\text{PWL}}^{ij}(\mathbf{x}, \mathbf{p}) \\ &= \sum_{r=1}^n (\mathbf{a}_{ij}^r \mathbf{p} + \mathbf{b}_{ij}^r) x_r + \mathbf{c}_{ij} \mathbf{p} + \mathbf{d}_{ij}, \quad (6) \\ \mathbf{x} &\in X_i, \mathbf{p} \in P_j, i \in I_x, j \in I_p, \end{aligned}$$

where x_r is the r th component of vector \mathbf{x} , X_i and P_j are non-overlapping simplices in X and P , respectively, such that $X = \bigcup_{i \in I_x} \bar{X}_i$ and $P = \bigcup_{j \in I_p} \bar{P}_j$ (\bar{X}_i

and \bar{P}_j denote the closure of X_i and P_j , respectively), and I_x and I_p are index sets of the cells in the state and parameter spaces, respectively. Also, $\mathbf{a}_{ij}^r, \mathbf{b}_{ij}^r, \mathbf{c}_{ij}$, and \mathbf{d}_{ij} are modeling parameters that must be calculated. In the compact form, Eq. (6) could be written as

$$\mathbf{f}_{\text{PWL}}^{ij}(\mathbf{x}, \mathbf{p}) = A_{ij}(\mathbf{p})\mathbf{x} + B_{ij}(\mathbf{p}), \quad \mathbf{x} \in X_i, \mathbf{p} \in P_j, \quad (7)$$

where the entries of $A_{ij}(\mathbf{p})$ and $B_{ij}(\mathbf{p})$ are affine functions of parameters.

4 Parameter dependent piecewise affine modeling

We define $X_i \times P_j = \{(\mathbf{x}, \mathbf{p}) | \mathbf{x} \in X_i, \mathbf{p} \in P_j\}$, $i \in I_x$ and $j \in I_p$, as multiplied simplex, where $X_i \subset X$ and $P_j \subset P$ are simplices. In this section, the modeling method is presented for one multiplied simplex $X_i \times P_j$, and then repeating this modeling for all multiplied simplices results in the approximated PD-PWA system. Let $\mathbf{f}(\mathbf{x}, \mathbf{p}): \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$, $X_i = \Delta(\mathbf{V}_{x0}^i, \mathbf{V}_{x1}^i, \dots, \mathbf{V}_{xn}^i)$ and $P_j = \Delta(\mathbf{V}_{p0}^j, \mathbf{V}_{p1}^j, \dots, \mathbf{V}_{pm}^j)$, where \mathbf{V}_{x0}^i and \mathbf{V}_{p0}^j are corners of X_i and P_j respectively, and

$$\begin{cases} \mathbf{V}_{xk}^i = \mathbf{V}_{x0}^i + \mathbf{U}_{xk}^i = \mathbf{V}_{x0}^i + \mathbf{D}_x \sum_{r=0}^k \mathbf{e}_r, & k = 0, 1, \dots, n, \\ \mathbf{V}_{ph}^j = \mathbf{V}_{p0}^j + \mathbf{U}_{ph}^j = \mathbf{V}_{p0}^j + \mathbf{D}_p \sum_{r=0}^h \mathbf{e}_r, & h = 0, 1, \dots, m. \end{cases} \quad (8)$$

Based on Eqs. (4) and (8) we have

$$\begin{cases} \lambda_{xk}^i = \mathbf{q}_{xk}^i (\mathbf{x} - \mathbf{V}_{x0}^i) + s_{xk}^i, & k = 0, 1, \dots, n, \\ \lambda_{ph}^j = \mathbf{q}_{ph}^j (\mathbf{p} - \mathbf{V}_{p0}^j) + s_{ph}^j, & h = 0, 1, \dots, m. \end{cases} \quad (9)$$

In Theorem 1, the modeling parameters for the PD-PWA model (6) are calculated.

Theorem 1 Consider $\mathbf{f}(\mathbf{x}, \mathbf{p}): \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ where $\mathbf{x} \in X_i$ and $\mathbf{p} \in P_j$. If for the PD-PWA system

$\mathbf{f}_{\text{PWL}}^{ij}(\mathbf{x}, \mathbf{p}) = \sum_{r=1}^n (\mathbf{a}_{ij}^r \mathbf{p} + \mathbf{b}_{ij}^r) x_r + \mathbf{c}_{ij} \mathbf{p} + \mathbf{d}_{ij}$, the modeling parameters are selected as

$$\begin{cases} a_{ij}^r = \sum_{h=0}^m \sum_{k=0}^n f(\mathbf{V}_{xk}^i, \mathbf{V}_{ph}^j) q_{xk,r}^i q_{ph}^j, \\ b_{ij}^r = \sum_{h=0}^m \sum_{k=0}^n f(\mathbf{V}_{xk}^i, \mathbf{V}_{ph}^j) (s_{ph}^j - q_{ph}^j \mathbf{V}_{p0}^j) q_{xk,r}^i, \\ c_{ij} = \sum_{h=0}^m \sum_{k=0}^n f(\mathbf{V}_{xk}^i, \mathbf{V}_{ph}^j) q_{ph}^j (s_{xk}^i - q_{xk}^i \mathbf{V}_{x0}^i), \\ d_{ij} = \sum_{h=0}^m \sum_{k=0}^n f(\mathbf{V}_{xk}^i, \mathbf{V}_{ph}^j) (s_{ph}^j - q_{ph}^j \mathbf{V}_{p0}^j) (s_{xk}^i - q_{xk}^i \mathbf{V}_{x0}^i), \end{cases} \quad (10)$$

$$\lambda_{ph}^j = \begin{cases} 1, & h = s, \\ 0, & h \neq s, \end{cases} \quad h = 0, 1, \dots, m.$$

Since

$$f_{\text{PWL}}^{ij}(\mathbf{x}, \mathbf{p}) = \sum_{h=0}^m \sum_{k=0}^n f(\mathbf{V}_{xk}^i, \mathbf{V}_{ph}^j) \lambda_{ph}^j \lambda_{xk}^i, \quad (11)$$

it is concluded that

$$\begin{aligned} f_{\text{PWL}}^{ij}(\mathbf{x}^*, \mathbf{p}^*) &= \sum_{h=0}^m \sum_{k=0}^n f(\mathbf{V}_{xk}^i, \mathbf{V}_{ph}^j) \lambda_{xk}^i \lambda_{ph}^j \\ &= f(\mathbf{V}_{x^i}, \mathbf{V}_{p^j}) = f(\mathbf{x}^*, \mathbf{p}^*). \end{aligned}$$

where $q_{xk,r}^i$ is the r th component of q_{xk}^i , then $f(\mathbf{x}, \mathbf{p}) = f_{\text{PWL}}^{ij}(\mathbf{x}, \mathbf{p})$ at the vertices of $X_i \times P_j$.

Proof By setting modeling parameters as Eq. (10) in PD-PWA function (6) and using Eq. (9), we have

$$\begin{aligned} f_{\text{PWL}}^{ij}(\mathbf{x}, \mathbf{p}) &= \sum_{r=1}^n (a_{ij}^r \mathbf{p} + b_{ij}^r) x_r + c_{ij} \mathbf{p} + d_{ij} \\ &= \sum_{r=1}^n \left(\sum_{h=0}^m \sum_{k=0}^n f(\mathbf{V}_{xk}^i, \mathbf{V}_{ph}^j) q_{xk,r}^i q_{ph}^j \mathbf{p} \right. \\ &\quad \left. + \sum_{h=0}^m \sum_{k=0}^n f(\mathbf{V}_{xk}^i, \mathbf{V}_{ph}^j) (s_{ph}^j - q_{ph}^j \mathbf{V}_{p0}^j) q_{xk,r}^i \right) x_r \\ &\quad + \sum_{h=0}^m \sum_{k=0}^n f(\mathbf{V}_{xk}^i, \mathbf{V}_{ph}^j) q_{ph}^j (s_{xk}^i - q_{xk}^i \mathbf{V}_{x0}^i) \mathbf{p} \\ &\quad + \sum_{h=0}^m \sum_{k=0}^n f(\mathbf{V}_{xk}^i, \mathbf{V}_{ph}^j) (s_{ph}^j - q_{ph}^j \mathbf{V}_{p0}^j) (s_{xk}^i - q_{xk}^i \mathbf{V}_{x0}^i) \\ &= \sum_{r=1}^n \left(\sum_{h=0}^m \sum_{k=0}^n f(\mathbf{V}_{xk}^i, \mathbf{V}_{ph}^j) (q_{ph}^j \mathbf{p} + s_{ph}^j - q_{ph}^j \mathbf{V}_{p0}^j) q_{xk,r}^i \right) x_r \\ &\quad + \sum_{h=0}^m \sum_{k=0}^n f(\mathbf{V}_{xk}^i, \mathbf{V}_{ph}^j) (q_{ph}^j \mathbf{p} + s_{ph}^j - q_{ph}^j \mathbf{V}_{p0}^j) (s_{xk}^i - q_{xk}^i \mathbf{V}_{x0}^i) \\ &= \sum_{h=0}^m \sum_{k=0}^n f(\mathbf{V}_{xk}^i, \mathbf{V}_{ph}^j) \lambda_{ph}^j q_{xk}^i \mathbf{x} \\ &\quad + \sum_{h=0}^m \sum_{k=0}^n f(\mathbf{V}_{xk}^i, \mathbf{V}_{ph}^j) \lambda_{ph}^j (s_{xk}^i - q_{xk}^i \mathbf{V}_{x0}^i) \\ &= \sum_{h=0}^m \sum_{k=0}^n f(\mathbf{V}_{xk}^i, \mathbf{V}_{ph}^j) \lambda_{ph}^j (q_{xk}^i \mathbf{x} + s_{xk}^i - q_{xk}^i \mathbf{V}_{x0}^i) \\ &= \sum_{h=0}^m \sum_{k=0}^n f(\mathbf{V}_{xk}^i, \mathbf{V}_{ph}^j) \lambda_{ph}^j \lambda_{xk}^i. \end{aligned}$$

By Eq. (1), for $\mathbf{x}^* = \mathbf{V}_{x^i}^i \in \{\mathbf{V}_{x0}^i, \mathbf{V}_{x1}^i, \dots, \mathbf{V}_{xn}^i\}$ and $\mathbf{p}^* = \mathbf{V}_{p^j}^j \in \{\mathbf{V}_{p0}^j, \mathbf{V}_{p1}^j, \dots, \mathbf{V}_{pm}^j\}$, we have

$$\lambda_{xk}^i = \begin{cases} 1, & k = t, \\ 0, & k \neq t, \end{cases} \quad k = 0, 1, \dots, n,$$

Therefore, with partitioning X and P into simplices separately, and calculating $f_{\text{PWL}}^{ij}(\mathbf{x}, \mathbf{p})$ for $X_i \times P_j$, $\forall i \in I_x$ and $\forall j \in I_p$, the total behavior of the PD-NL system is described by a state space model, which is affine in states and parameters separately.

Theorem 2 The approximated function $f_{\text{PWL}}(\mathbf{x}, \mathbf{p})$ is continuous on the boundaries of X_i , $i \in I_x$ and on the boundaries of P_j , $j \in I_p$.

Proof Supposing $\mathbf{p} \in P_j$ and $\mathbf{x} \in X_i \cap X_l$, we have

$$\begin{cases} f_{\text{PWL}}^{ij}(\mathbf{x}, \mathbf{p}) = \sum_{k=0}^m \sum_{r=0}^n \lambda_{pk}^j \lambda_{xr}^i f(\mathbf{V}_{xr}^i, \mathbf{V}_{pk}^j), & \mathbf{x} \in X_i, \mathbf{p} \in P_j, \\ f_{\text{PWL}}^{lj}(\mathbf{x}, \mathbf{p}) = \sum_{k=0}^m \sum_{r=0}^n \lambda_{pk}^j \lambda_{xr}^l f(\mathbf{V}_{xr}^l, \mathbf{V}_{pk}^j), & \mathbf{x} \in X_l, \mathbf{p} \in P_j. \end{cases} \quad (12)$$

Let V_{X_i} and V_{X_l} be the sets of vertices of X_i and X_l , respectively. Since $\mathbf{x} \in X_i \cap X_l$, this implies that $\mathbf{x} \in \Delta(V_{X_i} \cap V_{X_l})$. Based on Eq. (4), $\forall \mathbf{x} \in \text{conv}\{V_{X_i} \cap V_{X_l}\}$ where conv stands for the convex hull (Boyd *et al.*, 1994),

$$\forall r, r' \text{ where } \mathbf{V}_{xr}^i, \mathbf{V}_{xr'}^l \in V_{X_i} \cap V_{X_l} \text{ and } \mathbf{V}_{xr}^i = \mathbf{V}_{xr'}^l,$$

$$\lambda_{xr}^i = \lambda_{xr'}^l;$$

$$\forall r, r' \text{ where } \mathbf{V}_{xr}^i, \mathbf{V}_{xr'}^l \notin V_{X_i} \cap V_{X_l}, \lambda_{xr}^i = \lambda_{xr'}^l = 0.$$

Thus, Eq. (12) implies that for $\mathbf{x} \in X_i \cap X_l$, $f_{\text{PWL}}^{ij}(\mathbf{x}, \mathbf{p}) = f_{\text{PWL}}^{lj}(\mathbf{x}, \mathbf{p})$. It is the same when \mathbf{p} belongs to the boundaries of simplices.

Remark 1 Considering Eq. (11) for $\mathbf{x} \in X_i$ and $\mathbf{p} \in P_j$, $i \in I_x$ and $j \in I_p$, we have

$$\begin{aligned} & \left\| \mathbf{f}_{\text{PWL}}^{ij}(\mathbf{x}, \mathbf{p}) - \mathbf{f}(\mathbf{x}, \mathbf{p}) \right\| \\ & \leq \sum_{k=0}^n \sum_{h=0}^m \lambda_{xk}^i \lambda_{ph}^j \left\| \mathbf{f}(\mathbf{V}_{xk}^i, \mathbf{V}_{ph}^j) - \mathbf{f}(\mathbf{x}, \mathbf{p}) \right\| \end{aligned} \quad (13)$$

If $\mathbf{f}(\mathbf{x}, \mathbf{p})$ is a continuous function, it is possible to define

$$\varepsilon = \max_{\substack{X_i \subset X \\ P_j \subset P}} \max_{\substack{\mathbf{x}_0, \mathbf{x}_1 \in X_i \\ \mathbf{p}_0, \mathbf{p}_1 \in P_j}} \left\| \mathbf{f}(\mathbf{x}_0, \mathbf{p}_0) - \mathbf{f}(\mathbf{x}_1, \mathbf{p}_1) \right\|.$$

Thus,

$$\begin{aligned} & \left\| \mathbf{f}_{\text{PWL}}^{ij}(\mathbf{x}, \mathbf{p}) - \mathbf{f}(\mathbf{x}, \mathbf{p}) \right\| \\ & \leq \sum_{k=0}^n \sum_{h=0}^m \lambda_{xk}^i \lambda_{ph}^j \varepsilon = \varepsilon \sum_{k=0}^n \sum_{h=0}^m \lambda_{xk}^i \lambda_{ph}^j = \varepsilon, \\ & \quad i \in I_x, j \in I_p, \end{aligned}$$

which implies

$$\left\| \mathbf{f}_{\text{PWL}}(\mathbf{x}, \mathbf{p}) - \mathbf{f}(\mathbf{x}, \mathbf{p}) \right\| \leq \varepsilon.$$

X_i and P_j are non-overlapping simplices in X and P , respectively, such that $X = \bigcup_{i \in I_x} \bar{X}_i$ and $P = \bigcup_{j \in I_p} \bar{P}_j$.

So, any continuous PD-NL function can be approximated by a PD-PWA function with arbitrary accuracy, provided that the function domain is partitioned in a large enough number of sub-domains.

Remark 2 By expanding state space to $\mathbf{z} = [\mathbf{x}^T \mathbf{p}^T]^T$ and partitioning the new space into simplices, PD-PWA modeling has been done for PD-NL systems (Storace and de Feo, 2005a; 2005b; Storace and Bizzarri, 2007). It is referred to as modeling based on conventional simplices from now on. Here we want to compare the number of simplices and as a result the number of subsystems that constitute the overall PD-PWA system, for modeling based on conventional simplices and modeling based on multiplied simplices. Let $\mathbf{x} \in X \subset \mathbb{R}^n$, $\mathbf{p} \in P \subset \mathbb{R}^m$ and without loss of generality, suppose that the component of each dimension of \mathbf{x} and \mathbf{p} is partitioned into N subintervals or $N_{x_i} = N_{p_j} = N$, $\forall i \in I_x$ and $j \in I_p$. So, the number of hyper-rectangles and the number of conventional simplices are N^{n+m} and $\theta_C = N^{n+m} \times (n+m)!$, respectively. For the method proposed here, the number of simplices for the parameter space, the number of simplices for the state space, and the number of multiplied simplices are $N^m \times m!$, $N^n \times n!$, and $\theta_M = (N^m \times m!) \times (N^n \times n!) = N^{n+m} \times m! \times n!$, respectively.

Lemma 1 For $X \subset \mathbb{R}^n$ and $P \subset \mathbb{R}^m$, $\theta_C > \theta_M$.

Proof Since $n, m \neq 0$, we have

$$\frac{\theta_C}{\theta_M} = \frac{N^{n+m} (n+m)!}{N^{m+n} m! n!} = \binom{n+m}{n} = \binom{n+m}{m} > 1.$$

It can be seen that there is considerable difference between the number of multiplied simplices and the number of conventional simplices.

Remark 3 As mentioned before, there are two methods for representing a PWA system: conventional (Chien and Kuh, 1977) and canonical (Julian et al., 2000; Storace and de Feo, 2005a; 2005b; Bergami et al., 2006; Storace and Bizzarri, 2007). Here, based on conventional representation, the number of parameters that are necessary for representing an arbitrary PD-PWA function for modeling based on conventional simplices and modeling based on multiplied simplices are calculated. Consider $\mathbf{A}_C \mathbf{z} + \mathbf{B}_C$ as a PWA function on a conventional simplex. Since \mathbf{A}_C and \mathbf{B}_C are a matrix and a vector with $n \times (n+m)$ and n dimensions, respectively, for a conventional simplex the number of scalar parameters that must be calculated for modeling is $\eta_C = n \times (n+m+1)$. For a multiplied simplex, consider $\mathbf{A}_M(\mathbf{p})\mathbf{x} + \mathbf{B}_M(\mathbf{p})$ as a PWA function of \mathbf{x} where entries of \mathbf{A}_M and \mathbf{B}_M are affine functions of \mathbf{p} . \mathbf{A}_M and \mathbf{B}_M are a matrix and a vector with $n \times n$ and n dimensions, respectively. Since every affine function of \mathbf{p} needs $m+1$ scalar parameters, the number of scalar parameters that must be calculated for modeling based on multiplied simplices is $\eta_M = (n^2 + n) \times (m+1)$. For representing the PWA system over the operating region, modeling based on conventional simplices needs $\Psi_C = \theta_C \eta_C$ scalar parameters and modeling based on conventional simplices needs $\Psi_M = \theta_M \eta_M$ scalar parameters. Regardless of the difference between η_C and η_M , due to the extra difference between θ_C and θ_M , Ψ_M is generally less than Ψ_C . This comparison is done for a typical system in Example 2.

Lemma 2 For $X \subset \mathbb{R}^n$ and $P \subset \mathbb{R}^m$, $\Psi_C > \Psi_M$.

Proof From Remark 3 we have

$$\begin{aligned} \frac{\Psi_C}{\Psi_M} &= \frac{(n+m)!(n+m+1)}{n!m!(n+1)(m+1)} \\ &= \frac{1}{n+m+2} \times \frac{(n+m+2)!}{(n+1)!(m+1)!} \end{aligned}$$

By defining $\alpha=n+m+2$ and $\beta=m+1$,

$$\frac{\Psi_C}{\Psi_M} = \frac{1}{\alpha} \binom{\alpha}{\beta}.$$

Since for $\beta \geq 2$, $\binom{\alpha}{\beta} > \alpha$, we have $\frac{\Psi_C}{\Psi_M} > 1$.

Remark 4 If for any \mathbf{x} , $\mathbf{f}(\mathbf{x}, \mathbf{p})$ is affine with respect to the k th component of \mathbf{p} , i.e., p_k , then it is not needed to approximate the function with respect to p_k and thus $N_{p_k}=1$. It is the same when for any \mathbf{p} , $\mathbf{f}(\mathbf{x}, \mathbf{p})$ is affine with respect to some components of \mathbf{x} .

Remark 5 There are different methods for region configuration of the domain on which the system is defined (Tarela and Martinez, 1993; Ulbig et al., 2010). In this study, a modeling method based on partitioning the domain into similar cells is addressed. This remark summarizes PD-PWA modeling for a given PD-NL system in an algorithm:

1. Find a uniform rectangular grid for the domain of state variations and the domain of parameter variations separately.
2. Partition each hyper-rectangle in each domain into simplices.
3. For each multiplied simplex that is defined by a simplex from the state space and a simplex from the parameter space, find PD-PWA approximation.

5 Approximation error

If $\mathbf{f}(\mathbf{x}, \mathbf{p})$ has the Lipschitz continuity property, then it is possible to obtain some useful conditions relating the approximation error to the Lipschitz constant of $\mathbf{f}(\mathbf{x}, \mathbf{p})$.

Definition 2 A function $\mathbf{f}(\mathbf{x}, \mathbf{p}): X \times P \rightarrow \mathbb{R}$, $X \times P \subset \mathbb{R}^n \times \mathbb{R}^m$ is said to be Lipschitz continuous if it satisfies the condition

$$\|\mathbf{f}(\mathbf{x}, \mathbf{p}) - \mathbf{f}(\hat{\mathbf{x}}, \hat{\mathbf{p}})\| \leq L \|\mathbf{x} - \hat{\mathbf{x}}\| + L \|\mathbf{p} - \hat{\mathbf{p}}\|,$$

$$\forall \mathbf{x}, \hat{\mathbf{x}} \in X, \forall \mathbf{p}, \hat{\mathbf{p}} \in P.$$

$\|\cdot\|$ denotes the Euclidean norm and $L > 0$ is the Lipschitz constant where $L = \sup\{\|\nabla \mathbf{f}(\mathbf{x}, \mathbf{p})\|_\infty, \mathbf{x} \in X,$

$\mathbf{p} \in P\}$ and $\nabla \mathbf{f}(\mathbf{x}, \mathbf{p}) = \left[\frac{\partial \mathbf{f}}{\partial x_1}, \dots, \frac{\partial \mathbf{f}}{\partial x_n}, \frac{\partial \mathbf{f}}{\partial p_1}, \dots, \frac{\partial \mathbf{f}}{\partial p_m} \right]$ is the

gradient of the function $\mathbf{f}(\mathbf{x}, \mathbf{p})$. If $X \times P$ is partitioned into multiplied simplices, it is possible to define the Lipschitz constant $L^{ij} > 0$ for each multiplied simplex $X_i \times P_j$ such that

$$\|\mathbf{f}(\mathbf{x}, \mathbf{p}) - \mathbf{f}(\hat{\mathbf{x}}, \hat{\mathbf{p}})\| \leq L^{ij} \|\mathbf{x} - \hat{\mathbf{x}}\| + L^{ij} \|\mathbf{p} - \hat{\mathbf{p}}\|,$$

$$\forall \mathbf{x}, \hat{\mathbf{x}} \in X_i, \forall \mathbf{p}, \hat{\mathbf{p}} \in P_j,$$

where $L^{ij} = \sup\{\|\nabla \mathbf{f}(\mathbf{x}, \mathbf{p})\|_\infty, \mathbf{x} \in X_i, \mathbf{p} \in P_j\}$.

Let $\mathbf{f}(\mathbf{x}, \mathbf{p}) = [f_1(\mathbf{x}, \mathbf{p}), f_2(\mathbf{x}, \mathbf{p}), \dots, f_n(\mathbf{x}, \mathbf{p})]^T$ where $f_k(\mathbf{x}, \mathbf{p}): X \times P \rightarrow \mathbb{R}$, $X \times P \subset \mathbb{R}^n \times \mathbb{R}^m$ for $k=1, 2, \dots, n$. If f_k ($k=1, 2, \dots, n$) is Lipschitz continuous, then

$$\|f_k(\mathbf{x}, \mathbf{p}) - f_k(\hat{\mathbf{x}}, \hat{\mathbf{p}})\| \leq L_{ij}^k \|\mathbf{x} - \hat{\mathbf{x}}\| + L_{ij}^k \|\mathbf{p} - \hat{\mathbf{p}}\|,$$

$$\forall \mathbf{x}, \hat{\mathbf{x}} \in X_i \subset X, \forall \mathbf{p}, \hat{\mathbf{p}} \in P_j \subset P,$$

where $L_{ij}^k = \sup\{\|\nabla f_k(\mathbf{x}, \mathbf{p})\|_\infty, \mathbf{x} \in X_i, \mathbf{p} \in P_j\}$. Defining $L_{ij} = \max_{k=1,2,\dots,n} L_{ij}^k$ we have

$$\|\mathbf{f}(\mathbf{x}, \mathbf{p}) - \mathbf{f}(\hat{\mathbf{x}}, \hat{\mathbf{p}})\|_\infty \leq L_{ij} \|\mathbf{x} - \hat{\mathbf{x}}\| + L_{ij} \|\mathbf{p} - \hat{\mathbf{p}}\|,$$

$$\forall \mathbf{x}, \hat{\mathbf{x}} \in X_i, \forall \mathbf{p}, \hat{\mathbf{p}} \in P_j. \tag{14}$$

From Eq. (13) we obtain

$$\|\mathbf{f}_{\text{PWL}}^{ij}(\mathbf{x}, \mathbf{p}) - \mathbf{f}(\mathbf{x}, \mathbf{p})\|_\infty$$

$$\leq \sum_{k=0}^n \sum_{h=0}^m \lambda_{xk}^i \lambda_{ph}^j \|\mathbf{f}(\mathbf{V}_{xk}^i, \mathbf{V}_{ph}^j) - \mathbf{f}(\mathbf{x}, \mathbf{p})\|_\infty.$$

Using Eq. (14), it is concluded that

$$\|\mathbf{f}_{\text{PWL}}^{ij}(\mathbf{x}, \mathbf{p}) - \mathbf{f}(\mathbf{x}, \mathbf{p})\|_\infty$$

$$\leq \sum_{k=0}^n \sum_{h=0}^m \lambda_{xk}^i \lambda_{ph}^j L_{ij} \|\mathbf{V}_{xk}^i - \mathbf{x}\| + \sum_{k=0}^n \sum_{h=0}^m \lambda_{xk}^i \lambda_{ph}^j L_{ij} \|\mathbf{V}_{ph}^j - \mathbf{p}\|,$$

$$\forall \mathbf{x}, \hat{\mathbf{x}} \in X_i, \forall \mathbf{p}, \hat{\mathbf{p}} \in P_j,$$

which implies

$$\|\mathbf{f}_{\text{PWL}}^{ij}(\mathbf{x}, \mathbf{p}) - \mathbf{f}(\mathbf{x}, \mathbf{p})\|_\infty$$

$$\leq L_{ij} \max_{\substack{k=0,1,\dots,n \\ h=0,1,\dots,m}} \|\mathbf{V}_{xk}^i - \mathbf{x}\| + L_{ij} \max_{\substack{k=0,1,\dots,n \\ h=0,1,\dots,m}} \|\mathbf{V}_{ph}^j - \mathbf{p}\|,$$

$$\forall \mathbf{x}, \hat{\mathbf{x}} \in X_i, \forall \mathbf{p}, \hat{\mathbf{p}} \in P_j.$$

It is straightforward to show that $\max_{\substack{k=0,1,\dots,n \\ h=0,1,\dots,m}} \|(V_{xk}^i, V_{ph}^j) - (x, y)\|$ is equal to the distance between (V_{x0}^i, V_{p0}^j) and (V_{xn}^i, V_{pm}^j) . Thus,

$$\|f_{PWL}^{ij}(x, p) - f(x, p)\|_{\infty} \leq L_{ij} \sqrt{\text{tr} D_x^2 + \text{tr} D_p^2}. \quad (15)$$

Note that all the simplices are equal by construction, so the value of $\max \|(V_{xk}^i, V_{ph}^j) - (x, p)\|$ does not depend on the choice of the simplex.

If $L_{ij} \sqrt{\text{tr} D_x^2 + \text{tr} D_p^2} < \delta, \forall i \in I_x, j \in I_p$, then the PD-PWA modeling error for the system (5) is less than δ , or

$$\|f_{PWL}(x, p) - f(x, p)\|_{\infty} < \delta, \quad x \in X, p \in P.$$

By solving the optimization problem (16), the state and parameter spaces can be partitioned into the minimum number of multiplied simplices such that the modeling error remains less than a specified value δ .

$$\begin{aligned} & \min_{N_{x_i}, N_{p_j}} \theta_M \\ \text{s.t. } & L_{ij} \sqrt{\text{tr} D_x^2 + \text{tr} D_p^2} < \delta, \quad \forall i \in I_x, j \in I_p. \end{aligned} \quad (16)$$

6 Numerical examples

We apply the PD-PWA modeling method proposed in this paper on two PD-NL systems. In Example 1, we compare modeling based on conventional simplices with modeling based on multiplied simplices. It is verified that the systems admitting different attractors for different parameters can also be modeled via this method while they can preserve their quantitative and qualitative properties in appropriate partitioning of the operating region. Example 2 demonstrates the applicability of the proposed modeling method to a system with three states and two parameters. Also, in this example, the operating region is partitioned into the minimum number of simplices such that the pre-defined upper bound holds on the modeling error.

Example 1 The Bautin system is described as

$$\begin{cases} \dot{x}_1 = p_1 x_1 - x_2 - x_1(x_1^2 + x_2^2)(x_1^2 + x_2^2 - p_2), \\ \dot{x}_2 = p_1 x_2 + x_1 - x_2(x_1^2 + x_2^2)(x_1^2 + x_2^2 - p_2), \end{cases} \quad (17)$$

where $-1.5 \leq x_1, x_2 \leq 1.5$ and $-1 \leq p_1, p_2 \leq 1$ (Storace and de Feo, 2005a). The trajectory of the system is forced by the parameters p_1 and p_2 . In this example, system (17) is approximated based on the following partitions:

$$\begin{cases} (A_1) N_{x_1} = N_{x_2} = 4, N_{p_1} = N_{p_2} = 1, \\ (A_2) N_{x_1} = N_{x_2} = 6, N_{p_1} = N_{p_2} = 1, \\ (A_3) N_{x_1} = N_{x_2} = 9, N_{p_1} = N_{p_2} = 1, \end{cases} \quad (18)$$

$$\begin{cases} (B_1) N_{x_1} = N_{x_2} = 4, N_{p_1} = N_{p_2} = 2, \\ (B_2) N_{x_1} = N_{x_2} = 6, N_{p_1} = N_{p_2} = 2, \\ (B_3) N_{x_1} = N_{x_2} = 9, N_{p_1} = N_{p_2} = 3. \end{cases} \quad (19)$$

Modeling with A_1, A_2 , and A_3 is based on multiplied simplices and modeling with B_1, B_2 , and B_3 is based on conventional simplices (Storace and de Feo, 2005a).

The number of simplices and the number of scalar parameters that must be calculated for modeling are compared in Tables 1 and 2 for partitioning based on conventional and multiplied simplices, respectively.

Since the vector field of system (17) is affine with respect to parameters, based on Remark 4, $N_{p_1} = N_{p_2} = 1$. This is one of the reasons for considerable difference between Tables 1 and 2, but the main reason is due to defining multiplied simplices and separating the spaces of state and parameter as explained in Remark 2.

Table 1 θ_C and Ψ_C for modeling based on conventional simplices

Partitioning	θ_C	Ψ_C
B_1	1536	15 360
B_2	3456	34 560
B_3	17 496	174 960

Table 2 θ_M and Ψ_M for modeling based on multiplied simplices

Partitioning	θ_M	Ψ_M
A_1	64	1152
A_2	144	2592
A_3	324	5832

The comparison between the original system and approximated models is made for three different sets of parameters: $S_1=(-0.5, 0.5)$, $S_2=(-0.05, 0.5)$, and $S_3=(0.5, 0.5)$. Figs. 1 and 2 show the phase portraits corresponding to these parameter values and partitioning mentioned in Eqs. (18) and (19). Starting points were selected as the following pairs: $(\pm 1.4, \pm 1.4)$, $(0.25, 0.25)$, and $(-4, -4)$.

For more refined partition, the behavior of the approximated system is more similar to the original one from both quantitative and qualitative points of view, which can be seen by comparing the first columns of Figs. 1 and 2 with the third column of each. However, the approximated systems are obtained via fewer simplices and less needed memory for saving modeling parameters in the modeling method based on multiplied simplices.

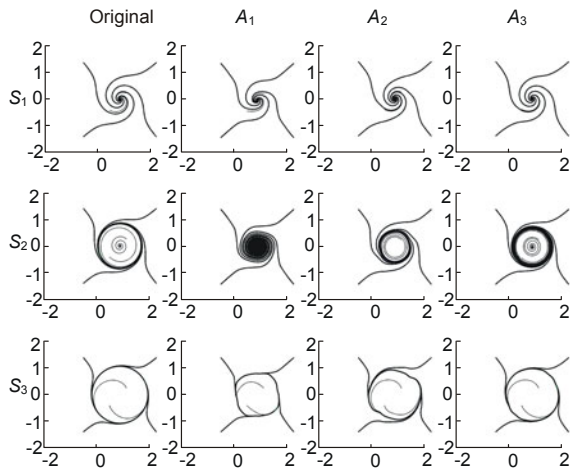


Fig. 1 Some phase portrait for the original system (17) and approximated systems based on A_1, A_2 , and A_3

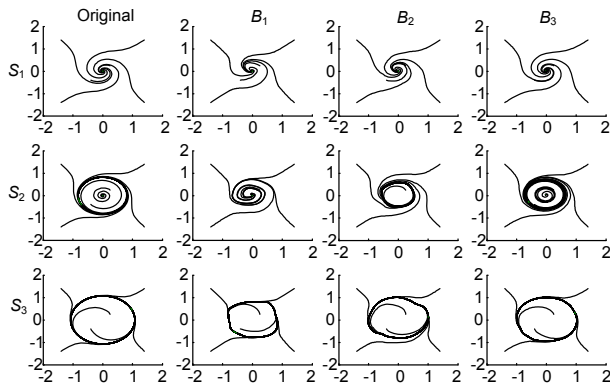


Fig. 2 Some phase portraits for the original system (17) and approximated systems based on B_1, B_2 , and B_3

Example 2 The Colpitts oscillator is described as (Maggio et al., 1999; de Feo et al., 2000)

$$\begin{cases} \dot{x}_1 = \frac{10^g}{Q(1-k)}[-\alpha_F(e^{-x_2} - 1) + x_3], \\ \dot{x}_2 = \frac{10^g}{Qk}[(1 - \alpha_F)(e^{-x_2} - 1) + x_3], \\ \dot{x}_3 = -\frac{Qk(1-k)}{10^g}(x_1 + x_2) - \frac{1}{Q}x_3. \end{cases} \quad (20)$$

The parameters are set as follows: $\alpha_F=0.996$, $k=0.5$, and g and Q vary in $[0.1, 0.5]$ and $[0.5, 2.5]$, respectively. Also, $-6 \leq x_1 \leq 6$, $-4 \leq x_2 \leq 4$, and $-2 \leq x_3 \leq 2$. For constant values of parameters g and Q , the dynamics of this oscillator is affine with respect to x_1 and x_3 .

Thus, based on Remark 4, $N_{x_1}=N_{x_3}=1$. To select N_{x_2} , N_g , and N_Q , the optimization problem (16) was solved via a genetic algorithm for three different values of δ . The results are shown in Table 3. For each δ , the partitioning was done with the smallest number of simplices such that the modeling error remains less than δ . The behaviors of the original system and the approximated systems, corresponding to the partitioning mentioned in Table 3, are shown in Fig. 3. The starting point is $(-1, 1, -1)$.

Table 3 Results of optimization problem (16) for different values of δ

δ	N_{x_2}	N_g	N_Q	θ_M
40	3	1	2	72
30	4	1	2	96
10	10	2	2	480

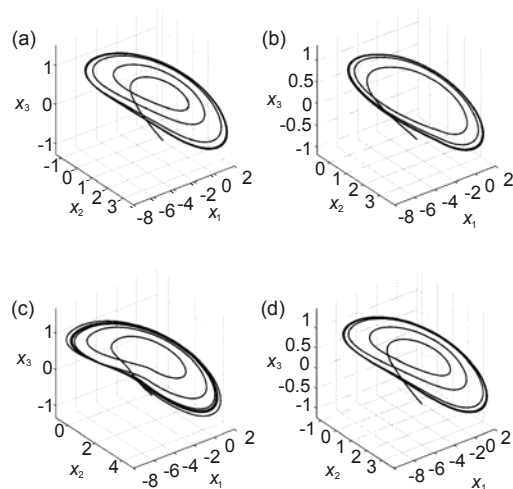


Fig. 3 State trajectories of the Colpitts oscillator (a) is the original system; (b), (c), (d) are approximated systems corresponding to $\delta=40, \delta=30, \delta=10$, respectively

7 Conclusions and further research

This work presents an automated PD-PWA modeling procedure for PD-NL systems. Using multiplied simplices is the key point for our method. It is shown that for approximating a PD-NL system, modeling based on multiplied simplices requires fewer subsystems compared to its similar modeling method. This reduction of the number of subsystems makes the analysis and control synthesis much easier. Also, simulation results show that the modeling method is able to follow the trajectories of the original system well. The main motivation of the approach is to find a method for stability analysis of PD-NL systems in a unified framework. This aspect and others, including performance analysis and control synthesis, are now under investigation.

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