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Enlarging the guaranteed region of attraction in nonlinear systems with bounded parametric uncertainty

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Abstract: A novel approach to enlarge the guaranteed region of attraction in nonlinear systems with bounded parametric uncertainties based on the design of a nonlinear controller is proposed. The robust domain of attraction (RDA) is estimated using the parameter-dependent quadratic Lyapunov function and enlarged by the optimal controlling parameters. The problem of extending the RDA is indicated in a form of three-layer optimization problem. Some examples illustrate the efficiency of the proposed strategy in enlarging RDA.

Key words: Lyapunov function (LF), Optimal controlling parameters, Robust domain of attraction (RDA)
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1 Introduction

Determining the domain of attraction (DA) of a stable equilibrium point is an important problem in nonlinear system theory. In general, DA cannot be exactly calculated. Different methods have been proposed to estimate DA. These methods can be classified into two general groups, Lyapunov-based and non-Lyapunov-based. The first group contains two main steps (Rapoport, 1999; Barreiro *et al.*, 2002; Rapoport and Morozov, 2008; Zhai *et al.*, 2009; Topcu *et al.*, 2010; Matallana *et al.*, 2011; Haghighatnia and Moghaddam, 2012): (1) A suitable Lyapunov function (LF) is suggested based on the structure of the system; (2) DA is estimated based on the suggested LF.

Lyapunov-based methods usually have some limitations, such as the dependence of the size of the estimated DA on the chosen LF, and usage of usual optimization tools such as the linear matrix inequality (LMI) or moment matrices which are usually applicable just for polynomial LFs (Li *et al.*, 2009). In spite of these constraints, the simplicity of these methods leads to their wide applications in estimating the DA.

Designing controllers to enlarge DA is another problem of interest. Bakhtiari and Yazdanpanah (2005) proposed an algorithm to design a nonlinear controller for polynomial systems, which maximizes the estimated DA using a predefined LF. The proposed method finds an optimal quadratic LF that maximizes the volume of DA via LMI-based convexification techniques. Chesi (2005) proposed a technique to compute static nonlinear output feedback controllers to enlarge the largest estimate of the DA defined by a given LF for polynomial systems. Specifically, the controller was supposed to be polynomial in the measurable output with powers and coefficients range a priori selectable. A lower bound of the maximum achievable estimate of the DA and a corresponding controller were obtained through LMI optimization. By exploiting known relaxation based on sum of squares of polynomials, the lower bound is computed through a generalized eigenvalue problem, i.e., a quasi-convex LMI optimization whose solution is efficiently calculated.

Real systems are often characterized by the presence of uncertain parameters, which cannot be measured exactly or are subject to variations. This means that DA is uncertain as well, since in general, it depends on such parameters. In such cases, one needs to consider the robust domain of attraction (RDA).

Calculating the actual RDA remains an unsolved problem; however, the following solutions are suggested: (1) estimating RDA via a parameter-dependent LF (Trofino, 2000; Chesi, 2004a); (2) finding a common LF to prove the robust local stability (Chesi, 2004a); (3) RDA estimation through a generalized Zubove method. These solutions all have limitations. Parameter-dependent LF is applicable only for time invariant uncertainties. In addition, there is no general LF structure. In most literature, quadratic LF is used leading to a conservative estimation of DA. Although RDA of systems with time varying uncertainty can be estimated through a common LF, finding such a common LF in general is impossible. In the third method, the viscosity solution of straightforward generalization of the classical Zubove equation is used to characterize the RDA of a nonlinear system with time varying perturbations (Camilli *et al.*, 2002). To solve Zubove's equation, the method of characteristic is used. This method requires the solution of the nonlinear system, and in fact the knowledge of DA, which is in general impossible (Kaslik *et al.*, 2005).

Some LMI methods based on sum of squares relaxations were proposed by Chesi (2009) for estimating the robust largest estimate of DA and LFs with a polynomial dependence on the state and with polynomial dependence on the uncertainty which is supposed to vary in a polytope. Camilli *et al.* (2001) provided a generalized Zubov method to uncertain systems. Paice and Wirth (1998) investigated the robustness of DA under time-varying perturbations and proposed an iterative algorithm that asymptotically gives the RDA. Trofino (2000), Chesi (2004b), and Tan (2006) considered parametric uncertainties. Chesi (2009) focused on computing the largest sublevel set of a given LF that can be certified to be an invariant subset of DA. Parameter-dependent LFs that lead to potentially less conservative results at the expense of increased computational complexity were proposed by Trofino (2000) and Tan (2006).

While the estimation of RDA has been largely addressed, enlarging the RDA has been less studied. Chesi (2011) designed a polynomial output controller that enlarges the DA of the equilibrium point of interest for all admissible uncertainties. For this problem, a strategy based on LMI optimizations and the square matrix representation of polynomials was proposed.

In this paper, a new approach to enlarge RDA in uncertain systems with bounded parametric uncertainty based on the design of controller is proposed. This method indicates an appropriate structure of nonlinear controller and finds optimal controlling parameters, such that they enlarge the intersection of sphere regions which are obtained from dependent quadratic LFs on parameters. The problem of enlarging the RDA is defined in the form of a novel three-layer optimization problem that focuses on extending RDA. The optimal controlling parameters are found from this optimization problem such that the eigenvalues of the Jacobian matrix of the dynamic system are forced to belong to the left half of the complex space and the RDA is enlarged. The DA is estimated as the controlling parameter and the uncertain parameter related function. This optimization problem finds the best controlling parameters which can effectively extend the RDA. Firstly, the RDA is found for a specific value controlling parameter, and then the best value of the controlling parameter which leads to the maximum radius of the RDA sphere shaped region is calculated.

2 Preliminaries

2.1 Definitions

Consider the following system:

$$\dot{x}_i = h_i(\boldsymbol{X}),\ i=1,2,...,n,\ \boldsymbol{X}_{\mathrm{e}}\in\mathbb{R}^n,\ \boldsymbol{X}(t_0)=\boldsymbol{X}_0. \quad (1)$$

Definition 1 (Equilibrium point) (Khalil, 2002) A point $\boldsymbol{X}_{\mathrm{e}}\in\mathbb{R}^n$ is called an equilibrium point of Eq. (1) if $\forall i\ h_i(\boldsymbol{X}_{\mathrm{e}})=0$. The equilibrium points of Eq. (1) correspond to the intersection of the nullclines of the system, meaning the curves given by $\boldsymbol{H}(\boldsymbol{X})=\boldsymbol{0}$, where $\boldsymbol{H}=[h_1, h_2, \ldots, h_n]^{\mathrm{T}}$.

In the sequel, without loss of generality, we assume that the equilibrium point under study coincides with the origin of the state space of $\mathbb{R}^n$, $\boldsymbol{X}_{\mathrm{e}}=\boldsymbol{0}$.

Definition 2 (Stability) (Hahn, 1967) Let $\boldsymbol{X}(t, \boldsymbol{X}_{\mathrm{e}})$ denote the solution of Eq. (1), which at the initial time

t_0 passes through the initial point $\boldsymbol{X}_e \in \mathbb{R}^n$. The origin is defined as stable if $\forall \varepsilon>0$ there exists a $\delta>0$, such that

$$\forall t \geq t_0,\ \|\boldsymbol{X}(t, \boldsymbol{X}_e)\| < \varepsilon \text{ is valid whenever } \|\boldsymbol{X}_e\| < \delta. \tag{2}$$

Definition 3 (Asymptotic stability) (Hahn, 1967) The origin is defined as asymptotically stable if: it is stable; there exists an $\eta>0$ having the property; $\lim_{t\to\infty} \boldsymbol{X}(t, \boldsymbol{X}_0) = \boldsymbol{0}$ whenever $\|\boldsymbol{X}_0\| < \eta$.

Definition 4 (Positive and negative definite functions) (La Salle and Lefschetz, 1961; Hahn, 1967) Let $\boldsymbol{D} \subseteq \mathbb{R}^n$. A function $V(\boldsymbol{X})$: $\boldsymbol{D} \to \mathbb{R}$ is positive definite (positive semidefinite) on D if $V(\boldsymbol{0})=0$ and $V(\boldsymbol{X})>0$ ($V(\boldsymbol{X})\geq 0$) $\forall \boldsymbol{X}\boldsymbol{D}\backslash\{\boldsymbol{0}\}$. $V(\boldsymbol{X})$ is negative definite (negative semidefinite) if $-V(\boldsymbol{X})$ is positive definite (positive semidefinite).

Definition 5 (Lyapunov function) (Khalil, 2002) Let $V(\boldsymbol{X})$ be a continuously differentiable real-valued function defined on a domain $\boldsymbol{D} \subset \mathbb{R}^n$ containing the equilibrium $\boldsymbol{X}_e=\boldsymbol{0}$. The function $V(\boldsymbol{X})$ is called an LF of equilibrium $\boldsymbol{X}=\boldsymbol{0}$ of Eq. (1) if the following conditions hold: $V(\boldsymbol{X})$ is positive definite on $\boldsymbol{D}$; the time derivative of $V(\boldsymbol{X})$ is along the trajectories of Eq. (1); $\dot{V}(\boldsymbol{X}) = \left(\frac{\partial V}{\partial \boldsymbol{X}}\right)^{\mathrm{T}} \boldsymbol{H}(\boldsymbol{X})$ is negative definite on $\boldsymbol{D}$.

Definition 6 (Domain of attraction) (Khalil, 2002) The DA of the origin is given by

$$\mathrm{DA} = \{\boldsymbol{X}_0 \in \mathbb{R}^n \mid \lim_{t\to\infty} \boldsymbol{X}(t, \boldsymbol{X}_0) = \boldsymbol{0}\}. \tag{3}$$

Definition 7 (Robust domain of attraction) Consider an uncertain nonlinear system with an isolated equilibrium state $\boldsymbol{X}_e$, of the following form:

$$\dot{\boldsymbol{X}} = \boldsymbol{H}(\boldsymbol{X}, \boldsymbol{\Theta}), \quad \boldsymbol{X} \in \mathbb{R}^n,$$
$$\boldsymbol{\Theta} = [\theta_1, \theta_2, \ldots, \theta_m], \quad \boldsymbol{\Theta} \in \boldsymbol{B} \subset \mathbb{R}^m, \quad \boldsymbol{X}(t_0) = \boldsymbol{X}_0, \tag{4}$$

where $\boldsymbol{\Theta}$ is the uncertain parameter vector, $\boldsymbol{B}$ is a bounded set in $\mathbb{R}^m$, and m is the number of uncertain parameters.

$$\mathrm{RDA} = \{\boldsymbol{X}_0 \in \mathbb{R}^n \mid \lim_{t\to\infty} \boldsymbol{X}(t, \boldsymbol{X}_0, \boldsymbol{\Theta}) = \boldsymbol{0} \quad \forall \boldsymbol{\Theta} \in \boldsymbol{B}\}. \tag{5}$$

In this paper, we just consider a class of nonlinear systems with an independent equilibrium point of uncertainty.

2.2 Theorems

Theorem 1 (Estimation of the domain of attraction) (Khalil, 2002) Let $V(\boldsymbol{X})$ be an LF for the equilibrium $\boldsymbol{X}=\boldsymbol{0}$ of Eq. (1).

Consider that $\frac{\mathrm{d}V(\boldsymbol{X})}{\mathrm{d}t}$ is negative definite in the region

$$S(\boldsymbol{0}) = \{\boldsymbol{X} \mid V(\boldsymbol{X}) \leq c, c > 0\}. \tag{6}$$

Hence, every trajectory initiated within region $S(\boldsymbol{0})$ tends to $\boldsymbol{X}=\boldsymbol{0}$ as time tends to infinity.

Theorem 2 (Jacobian's eigenvalues and local asymptotic stability) (Hahn, 1967) Let $\boldsymbol{A} = \left.\frac{\partial \boldsymbol{H}}{\partial \boldsymbol{X}(\boldsymbol{X})}\right\|_{\boldsymbol{X}=\boldsymbol{0}}$ be the Jacobian of Eq. (1) at the origin. Then the origin is asymptotically stable if all eigenvalues of $\boldsymbol{A}$ have negative real parts; the origin is unstable if one or more eigenvalues of $\boldsymbol{A}$ have positive real parts.

Moreover, assume that Ω_c is bounded and contains the origin. If $\dot{V}(\boldsymbol{x})$ is negative definite in Ω_c, then the origin is asymptotically stable and every solution in Ω_c tends to the origin as t converges to infinity.

Theorem 3 (Lyapunov identity) (Khalil, 2002) If the equilibrium $\boldsymbol{X}=\boldsymbol{0}$ of Eq. (1) is asymptotically stable, then there exists an LF of the quadratic type, $V(\boldsymbol{X})=\boldsymbol{X}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{X}$, where $\boldsymbol{P}$ is a positive definite matrix that can be calculated from the so-called Lyapunov identity:

$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} = -\boldsymbol{Q}. \tag{7}$$

A common choice is to set $\boldsymbol{Q}=\boldsymbol{I}$, where $\boldsymbol{I}$ is the identity matrix.

Theorem 4 Consider the following representation of Eq. (1):

$$\boldsymbol{H}(\boldsymbol{X})=\boldsymbol{A}\boldsymbol{X}+\boldsymbol{H}_1(\boldsymbol{X}),$$

where $\boldsymbol{H}_1(\boldsymbol{X})$ comprises the nonlinear part of function $\boldsymbol{H}(\boldsymbol{X})$. It can be shown that if the following condition holds (Vidyasagar, 1993)

$$\frac{\|\boldsymbol{H}_1(\boldsymbol{X})\|}{\|\boldsymbol{X}\|} \le \frac{\lambda_{\min}(\boldsymbol{Q})}{2\lambda_{\max}(\boldsymbol{P})} \quad \forall \boldsymbol{X} \in B_r, \tag{8}$$

$V(\boldsymbol{X})$ and its time derivatives are positive and negative definite respectively within the ball B_r of radius r. It is clear that the larger the ratio of $\lambda_{\min}(\boldsymbol{Q})/\left(2\lambda_{\max}(\boldsymbol{P})\right)$, the wider the possible choice of r.

3 Main method

Consider Eq. (1) with the controlling input $\boldsymbol{u}$ and the uncertain parameter vector $\boldsymbol{\Theta}$ as follows:

$$\dot{\boldsymbol{X}} = \boldsymbol{H}(\boldsymbol{X},\boldsymbol{\Theta}) + \boldsymbol{G}(\boldsymbol{X})\boldsymbol{u},\ \boldsymbol{X} \in \mathbb{R}^n,$$
$$\boldsymbol{\Theta} = [\theta_1, \theta_2, ..., \theta_m]^{\mathrm{T}},\ \boldsymbol{\Theta} \in \boldsymbol{B} \subset \mathbb{R}^m,\ \boldsymbol{X}(t_0) = \boldsymbol{X}_0, \tag{9}$$

where $\boldsymbol{G}(\boldsymbol{X})\boldsymbol{u} = \boldsymbol{A}_g(\boldsymbol{\Theta},\boldsymbol{K})\boldsymbol{X} + \boldsymbol{F}_g(\boldsymbol{X},\boldsymbol{\Theta},\boldsymbol{K})$, $\boldsymbol{H}(\boldsymbol{X},\boldsymbol{\Theta}) = \boldsymbol{A}_h(\boldsymbol{\Theta})\boldsymbol{X} + \boldsymbol{F}_h(\boldsymbol{X},\boldsymbol{\Theta})$, $\boldsymbol{A}_g$ and $\boldsymbol{A}_h$ express the linear part of $\boldsymbol{G}$ and $\boldsymbol{H}$, respectively, $\boldsymbol{F}_g$ and $\boldsymbol{F}_h$ denote their nonlinear components, and vector $\boldsymbol{K}$=[$\boldsymbol{K}_{\mathrm{l}}\boldsymbol{K}_{\mathrm{n}}$] contains $\boldsymbol{K}_{\mathrm{l}}$ and $\boldsymbol{K}_{\mathrm{n}}$, which are controlling parameters for the linear and nonlinear parts of the controller, respectively. Therefore, Eq. (9) can be shown as follows:

$$\boldsymbol{H}(\boldsymbol{x},\boldsymbol{\Theta}) + \boldsymbol{G}(\boldsymbol{x})\boldsymbol{u}$$
$$= \left(\boldsymbol{A}_g(\boldsymbol{\Theta},\boldsymbol{K}) + \boldsymbol{A}_h(\boldsymbol{\Theta})\right)\boldsymbol{x} + \left(\boldsymbol{F}_g(\boldsymbol{x},\boldsymbol{\Theta},\boldsymbol{K}) + \boldsymbol{F}_h(\boldsymbol{x},\boldsymbol{\Theta},\boldsymbol{K})\right).$$

Hence,

$$\boldsymbol{H}(\boldsymbol{X},\boldsymbol{\Theta}) + \boldsymbol{G}(\boldsymbol{X})\boldsymbol{u} = \boldsymbol{A}(\boldsymbol{\Theta},\boldsymbol{K})\boldsymbol{X} + \boldsymbol{F}(\boldsymbol{X},\boldsymbol{\Theta},\boldsymbol{K}), \tag{10}$$

where $\boldsymbol{F}(\boldsymbol{X}, \boldsymbol{\Theta}, \boldsymbol{K})$ indicates the nonlinear part of Eq. (9).

Proposition 1 Consider the quadratic LF below which depends on the uncertain parameters as follows:

$$V(\boldsymbol{X},\boldsymbol{\Theta}) = \boldsymbol{X}^{\mathrm{T}}\boldsymbol{P}(\boldsymbol{\Theta})\boldsymbol{X}. \tag{11}$$

It can be shown that if the following condition holds for a special vector $\boldsymbol{\Theta}$, then there is the ball B_r of radius r which is the estimate of DA:

$$\frac{\|\boldsymbol{F}(\boldsymbol{X},\boldsymbol{\Theta})\|}{\|\boldsymbol{X}\|} \le \frac{\lambda_{\min}(\boldsymbol{Q})}{2\lambda_{\max}\left(\boldsymbol{P}(\boldsymbol{\Theta})\right)} \quad \forall \boldsymbol{X} \in B_r. \tag{12}$$

Proof The proof is clear considering Theorem 4 where $\boldsymbol{H}_1$ is replaced with $\boldsymbol{F}$, $\boldsymbol{P}$ is replaced with $\boldsymbol{P}(\boldsymbol{\Theta})$, and $\boldsymbol{\Theta}$ is a determined vector.

Proposition 2 The RDA of nonlinear Eq. (9) with the uncertain parameter vector $\boldsymbol{\Theta}$ is as follows:

$$\mathrm{RDA} = \bigcap_{\boldsymbol{\Theta}\in\boldsymbol{B}} B_r(\boldsymbol{\Theta}). \tag{13}$$

Proof According to Proposition 1 for a special vector of $\boldsymbol{\Theta}$, $B_r(\boldsymbol{\Theta})$ obtained from Eq. (12) is an estimate of DA, which means $B_r(\boldsymbol{\Theta})\subset$DA $\forall\boldsymbol{\Theta}\in\boldsymbol{B}$. In addition, considering Definition 7, RDA= $\{\boldsymbol{X}(\boldsymbol{0}) \mid \lim_{t\to\infty}\boldsymbol{X}(t,\boldsymbol{X}_0,\boldsymbol{\Theta}) \to \boldsymbol{X}_{\mathrm{e}}\ \ \forall\boldsymbol{\Theta}\in\boldsymbol{B}\}$, so one can imply that RDA=$\cap B_r(\boldsymbol{\Theta})$ $\forall\boldsymbol{\Theta}\in\boldsymbol{B}$ which is the largest region with guaranteed stability for every $\boldsymbol{\Theta}\in\boldsymbol{B}$.

Proposition 3 The RDA can be enlarged by choosing the appropriate values of controlling parameters.

Proof Considering Eqs. (12) and (13), the larger estimate of $B_r(\boldsymbol{\Theta})$ for each $\boldsymbol{\Theta}$ vector leads to the larger estimated RDA. Thus, choosing the controlling parameters through the following steps leads to the extended RDA (ERDA).

Considering RDA of Eq. (10), it is estimated with intersection of the spheres obtained using Lyapunov quadratic functions. Lyapunov funtions are dependent on the controlling parameters vector and the uncertain parameters as follows:

$$V(\boldsymbol{X},\boldsymbol{\Theta},\boldsymbol{K}) = \boldsymbol{X}^{\mathrm{T}}\boldsymbol{P}(\boldsymbol{\Theta},\boldsymbol{K})\boldsymbol{X}, \tag{14}$$

where $\boldsymbol{P}$ is positive definite, and as $\boldsymbol{P}$ is calculated from Eq. (15) and $\boldsymbol{A}$ in Eq. (15) is a function of $\boldsymbol{\Theta}$ and $\boldsymbol{K}$, solving Eq. (15) leads to a matrix $\boldsymbol{P}$ which is dependent on $\boldsymbol{K}$ and $\boldsymbol{\Theta}$. Thus, in Eq. (14), we indicate $\boldsymbol{P}$ as a function of $\boldsymbol{\Theta}$ and $\boldsymbol{K}$.

$$\boldsymbol{A}^{\mathrm{T}}(\boldsymbol{\Theta},\boldsymbol{K})\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}(\boldsymbol{\Theta},\boldsymbol{K}) = -\boldsymbol{Q}, \tag{15}$$

where $\boldsymbol{Q}$ is supposed to be an arbitrary positive definite matrix. The time derivative of the quadratic LF along the system's trajectory is defined as

$$\begin{aligned}&\dot{V}(\boldsymbol{X},\boldsymbol{\Theta},\boldsymbol{K})\\&=\dot{\boldsymbol{X}}\left(\boldsymbol{A}(\boldsymbol{X},\boldsymbol{\Theta},\boldsymbol{K})^{\mathrm{T}}\boldsymbol{P}(\boldsymbol{\Theta},\boldsymbol{K})\right.\\&\quad\left.+\boldsymbol{P}(\boldsymbol{\Theta},\boldsymbol{K})\boldsymbol{A}(\boldsymbol{X},\boldsymbol{\Theta},\boldsymbol{K})\right)\boldsymbol{X}\\&\quad+2\boldsymbol{X}^{\mathrm{T}}\boldsymbol{P}(\boldsymbol{\Theta},\boldsymbol{K})\boldsymbol{F}(\boldsymbol{X},\boldsymbol{\Theta},\boldsymbol{K}).\end{aligned}\tag{16a}$$

Since

$$-\boldsymbol{x}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{x}\le-\lambda_{\min}(\boldsymbol{Q})\|\boldsymbol{X}\|^2,\tag{16b}$$

$$\begin{aligned}&\left|\boldsymbol{X}^{\mathrm{T}}\boldsymbol{P}(\boldsymbol{\Theta},\boldsymbol{K})\boldsymbol{F}(\boldsymbol{X},\boldsymbol{\Theta},\boldsymbol{K})\right|\\&\quad\le\lambda_{\max}\left(\boldsymbol{P}(\boldsymbol{\Theta},\boldsymbol{K})\right)\|\boldsymbol{X}\|\cdot\|\boldsymbol{F}(\boldsymbol{X},\boldsymbol{\Theta},\boldsymbol{K})\|,\end{aligned}\tag{16c}$$

we have

$$\begin{aligned}&\dot{V}(\boldsymbol{X},\boldsymbol{\Theta},\boldsymbol{K})\\&\le-\lambda_{\min}(\boldsymbol{Q})\|\boldsymbol{X}\|^2+2\lambda_{\max}(\boldsymbol{P})\|\boldsymbol{X}\|\cdot\|\boldsymbol{F}(\boldsymbol{X},\boldsymbol{\Theta},\boldsymbol{K})\|\\&=\|\boldsymbol{X}\|\left[2\lambda_{\max}\left(\boldsymbol{P}(\boldsymbol{\Theta},\boldsymbol{K})\right)\|\boldsymbol{F}(\boldsymbol{X},\boldsymbol{\Theta},\boldsymbol{K})\|-\lambda_{\min}(\boldsymbol{Q})\|\boldsymbol{X}\|\right].\end{aligned}$$

To obtain $\dot{V}<0,$ it is sufficient to choose r>0 as

$$\frac{\|\boldsymbol{F}(\boldsymbol{X},\boldsymbol{\Theta},\boldsymbol{K})\|}{\|\boldsymbol{X}\|}\le\frac{\lambda_{\min}(\boldsymbol{Q})}{2\lambda_{\max}\left(\boldsymbol{P}(\boldsymbol{\Theta},\boldsymbol{K})\right)}\quad\forall\boldsymbol{X}\in B_r.$$

It is clear that the RDA can be enlarged by choosing appropriate values of controlling parameters.

According to Propositions 1–3, the following three-layer optimization algorithm can be employed to find the best values of controlling parameters which extend RDA. According to Eq. (6), the larger level set of $V(\boldsymbol{X}, \boldsymbol{\Theta}, \boldsymbol{K})$ leads to the better estimated DA. To find the maximum level set of LF which is fully contained in the region of negative definiteness of $\frac{\mathrm{d}V}{\mathrm{d}t}$, a single point in the state space should be found. This point corresponds to a tangential contact of level sets $V(\boldsymbol{X}, \boldsymbol{\Theta}, \boldsymbol{K})=c$ and $\frac{\mathrm{d}V}{\mathrm{d}t}$. In other words, the largest level set which satisfies the condition of Theorem 1 turns to be the smallest sphere contained in $\frac{\mathrm{d}V}{\mathrm{d}t}$=0. If LF is considered as Eq. (11), finding the largest level set for estimating DA can be reformulated as the third layer of Eq. (17). The desired solution of the third layer in Eq. (17) is also a single point in the state space, which corresponds to a contact of the ball B_r of radius r and the surface.

$$\frac{\|\boldsymbol{F}(\boldsymbol{X},\boldsymbol{\Theta},\boldsymbol{K})\|}{\|\boldsymbol{X}\|}-\frac{\lambda_{\min}(\boldsymbol{Q})}{2\lambda_{\max}\left(\boldsymbol{P}(\boldsymbol{\Theta},\boldsymbol{K})\right)}=0.$$

In the second layer, the intersection of spheres obtained from quadratic LFs dependent on uncertain parameters is considered as RDA. Finally, to calculate ERDA in the outer layer, the optimal controlling parameters are found. Thus, the three-layer optimization problem is as follows:

$$\text{1st}\begin{cases}R_{\max}=\max\limits_{K,R_k}R_k,\\\mathrm{Re}\{\lambda[\boldsymbol{A}(\boldsymbol{X}_{\mathrm{e}},\boldsymbol{\Theta},\boldsymbol{K})]\}<0,\\R_k=\min\limits_{\boldsymbol{\Theta},R_{\theta k}}(R_{\theta k}),\\\text{2nd}\begin{cases}R_{\theta k}=\min\limits_{R_\theta,\boldsymbol{X},\boldsymbol{P}}R_\theta,\\\text{3rd}\begin{cases}\|\boldsymbol{X}\|-R_\theta=0,\\\boldsymbol{A}(\boldsymbol{X}_{\mathrm{e}},\boldsymbol{\Theta},\boldsymbol{K})^{\mathrm{T}}\boldsymbol{P}(\boldsymbol{\Theta},\boldsymbol{K})\\\quad+\boldsymbol{P}(\boldsymbol{\Theta},\boldsymbol{K})\boldsymbol{A}(\boldsymbol{X}_{\mathrm{e}},\boldsymbol{\Theta},\boldsymbol{K})=-\boldsymbol{Q},\\\dfrac{\|\boldsymbol{F}(\boldsymbol{X},\boldsymbol{\Theta},\boldsymbol{K})\|}{\|\boldsymbol{X}\|}-\dfrac{\lambda_{\min}(\boldsymbol{Q})}{2\lambda_{\max}\left(\boldsymbol{P}(\boldsymbol{\Theta},\boldsymbol{K})\right)}=0.\end{cases}\end{cases}\end{cases}\tag{17}$$

Note that the constraints of the third layer in Eq. (17) may have many local solutions. To avoid dummy solutions, they have to be solved to global optimality; therefore, in this contribution, a standard implementation of a genetic algorithm is employed.

To clarify, the proposed algorithm for enlarging RDA is shown in Fig. 1. To decrease the complexity of Fig. 1, we consider $\boldsymbol{\Theta}=\theta$ and $\boldsymbol{K}=k$, but it is obvious that without loss of generality, it can be extended for arbitrary $\boldsymbol{\Theta}$ and $\boldsymbol{K}$ vectors.

According to Proposition 3, we propose a nonlinear controller with the following structure:

$$\boldsymbol{u}=\boldsymbol{K}_{\mathrm{n}}\boldsymbol{G}(\boldsymbol{X},\boldsymbol{\Theta})+\boldsymbol{K}_{\mathrm{l}}\boldsymbol{X},\tag{18}$$

where $\boldsymbol{K}_{\mathrm{n}}\boldsymbol{G}$ defines the nonlinear part of the controller, $\boldsymbol{K}_{\mathrm{l}}\boldsymbol{X}$ is the linear part, $\boldsymbol{K}_{\mathrm{n}}$ and $\boldsymbol{K}_{\mathrm{l}}$ are the controlling parameter vectors, and every element of $\boldsymbol{K}$ is bounded in [−1, 1]. The nonlinear structure of $\boldsymbol{G}$ is similar to $\boldsymbol{F}$.

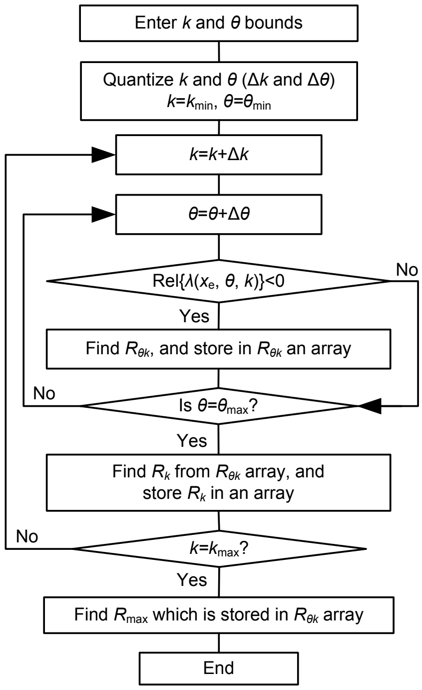

Fig. 1 The proposed algorithm for enlarging robust domain of attraction (RDA) for $\Theta=\theta$ and $K=k$

In the first layer of the optimization problem (17), we find the set of all admissible controlling parameters which lead to a Hurwitz matrix $\boldsymbol{A}$ and the optimal controlling parameters which lead to the ERDA.

4 Examples

In this section, we apply the proposed method on two uncertain nonlinear systems. The solution results show that the three-layer optimization algorithm implies an effective enlargement of RDA by calculating optimal values of controlling parameters.

4.1 Example 1

Consider the following nonlinear system:

$$\dot{x}_1 = -x_2, \quad \dot{x}_2 = x_1 - \theta(1-x_1^2)x_2 + u,$$

where θ is the controlling parameter and $\theta\in[1, 3]$. The analyzed equilibrium is (0, 0). The proposed structure of the desired controller is as follows:

$$u = k_n\theta(1-x_1^2)x_2 + k_1x_1,$$

where the proposed controller has the similar structure to that of the nonlinear system.

Using the optimization algorithm in Eq. (16), the optimal value of controlling parameters is obtained as $\boldsymbol{K}$=[0.8, 0.2], and the related $R_{\max}$, which is the radius of the ERDA, is 0.9689. As shown in Fig. 2, choosing such controlling parameters leads to a significant increase in the radius of the ERDA. In the absence of controllers, the radius of the estimated RDA for the nonlinear system is 0.4671. The controlling parameters are quantized with the step size equal to 0.1 and the uncertain parameters are quantized with the step size equal to 0.01.

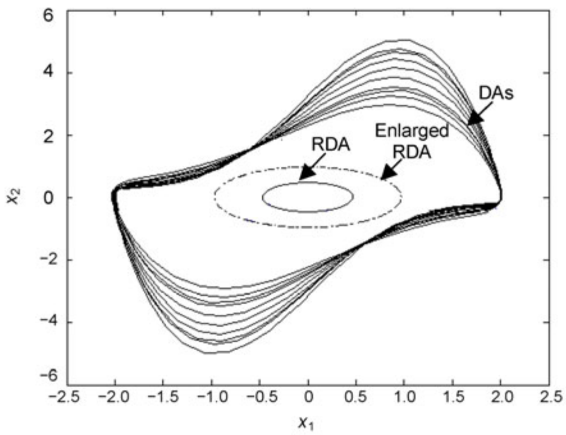

Fig. 2 RDA, enlarged RDA, and DAs of the van der Pol oscillator for different values of uncertain parameters without the controller

4.2 Example 2

In this example, we consider a torque controlled unstable pendulum with viscous friction (Cruck *et al.*, 2001), and try to stabilize such a system using an appropriate controlling parameter. The structure of the uncertain system is as follows:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = \sin x_1 - x_2 + u + \theta,$$

where θ is the controlling parameter and $\theta\in[-0.05, 0.05]$. The analyzed equilibrium is (0, 0). According to Proposition 2, the desired controller that has a structure similar to the structure of the nonlinear system is as follows:

$$u = k_g(-\sin x_1 - \theta) + k_1x_1.$$

In this example, in the absence of controllers and the presence of uncertainty, the system is unstable. Using the proposed controller stabilizes the system because the controlling parameters are designed such that the constrain in the first layer is satisfied. This inequality represents the asymptotic stability constraint of the equilibrium point $\boldsymbol{X}_e$ by imposing that the real part of the eigenvalues of the Jacobian matrix $\boldsymbol{A}$ be strictly negative.

In Fig. 3, the approximated DA in the absence of uncertainty is illustrated by solid curves. The system considered in Example 2 becomes unstable when there are uncertainties (Cruck *et al*., 2001). In this paper, we apply the three-layer algorithm to stabilize the nonlinear uncertain systems.

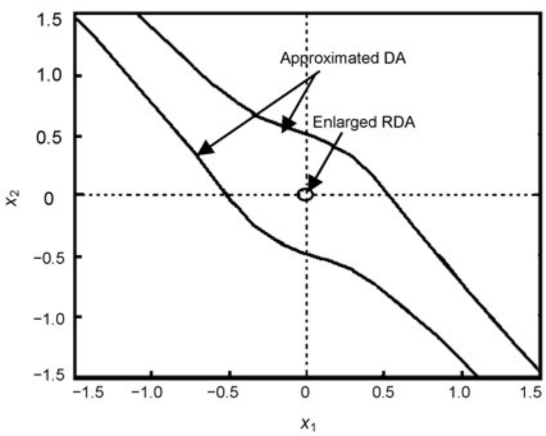

Fig. 3 Approximation of the domain of attraction of pendulum problem without uncertainty (Cruck *et al.*, 2001) and the estimated RDA of the pendulum problem

Using the optimization algorithm in Eq. (17), the optimal value of the controlling vector is obtained as $\boldsymbol{K}$=[1, −0.5], and the related R_{max}, which is the radius of extended RDA, is 0.0434. In absence of controllers, the system is unstable. The controlling parameters are quantized with the step size equal to 0.1 and the uncertain parameters are quantized with the step size equal to 0.01.

5 Conclusions

To enlarge RDA in uncertain systems, a new approach based on the design of a nonlinear controller is proposed in this paper. A three-layer optimization problem finds the optimal controlling parameters of this nonlinear controller to extend RDA. In the third layer of the optimization problem, the largest estimated DA is found. In the second layer, intersection of DAs that are dependent on uncertain parameters is obtained. Finally, in the first layer, the optimal controlling parameters leading to the largest RDA are found. The structure of the proposed nonlinear controller is similar to the structure of the dynamic system. The efficiency of the proposed methods is shown via simulations.

References

Bakhtiari, R., Yazdanpanah, M.J., 2005. Designing a Linear Controller for Polynomial Systems with the Largest Domain of Attraction via LMIs. Int. Conf. on Control and Automation, p.449-453. [doi:10.1109/ICCA.2005.1528161]

Barreiro, A., Aracil, J., Pagano, D., 2002. Detection of attraction domains of non-linear systems using bifurcation analysis and Lyapunov functions. *Int. J. Control*, **75**(5):314-327. [doi:10.1080/00207170110110568]

Camilli, F., Grune, L., Wirth, F., 2001. A generalization of Zubov's method to perturbed systems. *SIAM J. Control Optim.*, **40**(2):496-515. [doi:10.1137/S036301299936316X]

Camilli, F., Grune, L., Wirth, F., 2002. A Generalization of Zubove's Method to Perturbed Systems. Proc. 41st IEEE Conf. on Decision and Control, p.3518-3523. [doi:10.1109/CDC.2002.1184420]

Chesi, G., 2004a. On the Estimation of the Domain of Attraction for Uncertain Polynomial Systems via LMIs. Proc. 43rd IEEE Conf. on Decision and Control, p.881-886. [doi:10.1109/CDC.2004.1428796]

Chesi, G., 2004b. Estimating the domain of attraction for uncertain polynomial systems. *Automatica*, **40**(11):1981-1986. [doi:10.1016/j.automatica.2004.06.014]

Chesi, G., 2005. Computing output feedback controllers to enlarge the domain of attraction in polynomial systems. *IEEE Trans. Autom. Control*, **49**(10):1846-1850. [doi:10.1109/TAC.2004.835589]

Chesi, G., 2009. Estimating the domain of attraction for non-polynomial systems via LMI optimizations. *Automatica*, **45**(6):1536-1541. [doi:10.1016/j.automatica.2009.02.011]

Chesi, G., 2011. Control Synthesis for Enlarging the Robust Domain of Attraction in Uncertain Polynomial Systems. Proc. IASTED Int. Conf. on Control and Applications, p.729-738. [doi:10.2316/P.2011.729-038]

Cruck, E., Moitie, R., Seube, N., 2001. Estimation of basins of attraction for uncertain systems with affine and Lipschitz dynamics. *Dyn. Control*, **11**(3):211-227. [doi:10.1023/A:1015244102061]

Haghighatnia, S., Moghaddam, R.K., 2012. Directional extension of the domain of attraction to increase critical clearing time of nonlinear systems. *J. Am. Sci.*, **8**(1):444-449.

Hahn, W., 1967. Stability of Motion. Springer, Berlin (in

German).
Kaslik, E., Balint, A.M., Balint, S., 2005. Methods for determination and approximation of the domain of attraction. *Nonl. Anal. Theory Methods Appl.*, **60**(4):703-717. [doi:10.1016/j.na.2004.09.046]
Khalil, H.K., 2002. Nonlinear Systems (3rd Ed.). Prentice Hall, Upper Saddle River, New Jersey.
La Salle, J., Lefschetz, S., 1961. Stability by Liapunov's Direct Method. Academic Press, New York.
Li, C., Ryoo, C.S., Li, N., Cao, L., 2009. Estimating the domain of attraction via moment matrices. *Bull. Kor. Math. Soc.*, **46**(6):1237-1248. [doi:10.4134/BKMS.2009.46.6.1237]
Matallana, L.G., Blanco, A.M., Bandoni, J.A., 2011. Nonlinear dynamic systems design based on the optimization of the domain of attraction. *Math. Comput. Model.*, **53**(5-6): 731-745. [doi:10.1016/j.mcm.2010.10.011]
Paice, A., Wirth, F., 1998. Robustness Analysis of Domains of Attraction of Nonlinear Systems. Proc. Mathematical Theory of Networks and Systems, p.353-356.
Rapoport, L.B., 1999. Estimation of an Attraction Domain for Multivariable Lur'e Systems Using Looseless Extension of the S-Procedure. Proc. American Control Conf., p.2395-2396. [doi:10.1109/ACC.1999.786474]
Rapoport, L.B., Morozov, Y.V., 2008. Estimating the attraction domain of the invariant set in the problem of wheeled robot control. *Autom. Remote Control*, **69**(11):1859-1872. [doi:10.1134/S0005117908110039]
Tan, W., 2006. Nonlinear Control Analysis and Synthesis Using Sum-of-Squares Programming. PhD Thesis, University of California, Berkeley.
Topcu, U., Packard, A.K., Seiler, P., Balas, G.J., 2010. Robust region of attraction estimation. *IEEE Trans. Autom. Control*, **55**(1):137-142. [doi:10.1109/TAC.2009.2033751]
Trofino, A., 2000. Robust Stability and Domain of Attraction of Uncertain Nonlinear Systems. Proc. American Control Conf., p.3707-3710. [doi:10.1109/ACC.2000.879262]
Vidyasagar, M., 1993. Nonlinear Systems Analysis. Prentice Hall, New Jersey.
Zhai, G., Matsune, I., Imae, J., Kobayashi, T., 2009. A note on multiple Lyapunov functions and stability condition for switched and hybrid systems. *Int. J. Innov. Comput. Inf. Control*, **5**(5):1189-1199.