Journal of Zhejiang University-SCIENCE C (Computers & Electronics)
ISSN 1869-1951 (Print); ISSN 1869-196X (Online)
www.zju.edu.cn/jzus; www.springerlink.com
E-mail: jzus@zju.edu.cn

Hypothesis testing for reliability with a three-parameter Weibull distribution using minimum weighted relative entropy norm and bootstrap*

Xin-tao XIA†1, Yin-ping JIN[1], Yong-zhi XU[2], Yan-tao SHANG[1], Long CHEN[1]

([1]School of Mechatronical Engineering, Henan University of Science and Technology, Luoyang 471003, China)

([2]School of Mechatronical Engineering, Northwestern Polytechnical University, Xi'an 710072, China)

†E-mail: xiaxt1957@163.com; xiaxt@mail.haust.edu.cn

Received Sept. 19, 2012; Revision accepted Jan. 9, 2013; Crosschecked Jan. 10, 2013

Abstract: With the help of relative entropy theory, norm theory, and bootstrap methodology, a new hypothesis testing method is proposed to verify reliability with a three-parameter Weibull distribution. Based on the relative difference information of the experimental value vector to the theoretical value vector of reliability, six criteria of the minimum weighted relative entropy norm are established to extract the optimal information vector of the Weibull parameters in the reliability experiment of product lifetime. The rejection region used in the hypothesis testing is deduced via the area of intersection set of the estimated truth-value function and its confidence interval function of the three-parameter Weibull distribution. The case studies of simulation lifetime, helicopter component failure, and ceramic material failure indicate that the proposed method is able to reflect the practical situation of the reliability experiment.

Key words: Reliability, Hypothesis testing, Three-parameter Weibull distribution, Weighted relative entropy, Norm, Bootstrap
doi:10.1631/jzus.C12a0241 **Document code:** A **CLC number:** TB114.3

1 Introduction

For many products, such as spacecraft, airplanes, bullet trains, nuclear reactors, and submarines, reliability is regarded as an important indicator for their safe and stable operation (Johnson, 1970; Pierce *et al.*, 2011; Tan *et al.*, 2011; Chatterjee and Bandopadhyay, 2012; Lee and Pan, 2012; Li *et al.*, 2012; Niesłony *et al.*, 2012; Prawoto and Dillon, 2012; Zhang *et al.*, 2012). In reliability tests and analyses, a three-parameter Weibull distribution has been widely used in scientific studies and engineering practices (Jiang and Zuo, 1999; Deng *et al.*, 2004; Zhao *et al.*, 2010; Abbasi *et al.*, 2011; Bartkute-Norkuniene and Sakalauskas, 2011; El-Adll, 2011; Toasa Caiza and Ummenhofer, 2011; Qian, 2012).

So far, the primary concern is to evaluate the Weibull parameters, and many good results have been achieved. The main research methods include the fitting method (Kaplan and Meier, 1958; Johnson, 1970; Nelson, 1990; Duffy *et al.*, 1993; Luxhoj and Shyur, 1995), the variable neighborhood search and simulated annealing method (Abbasi *et al.*, 2011), the random number method (El-Adll, 2011), the maximum likelihood method (Harris, 1991; Jiang and Zuo, 1999; Qian, 2012), and the moment and probability weighted moment method (Deng *et al.*, 2004; Zhao *et al.*, 2010; Bartkute-Norkuniene and Sakalauskas, 2011; Toasa Caiza and Ummenhofer, 2011; Shafiee-zadeh and Ellingwood, 2012).

From a standpoint of uncertainty theory, the result of the reliability experiment generally contains uncertainties, based on the fact that many complex and variable factors appear in life tests, and may therefore be distorted, resulting in hidden dangers. Thus, hypothesis testing for reliability becomes a new

* Project (Nos. 51075123 and 50675011) supported by the National Natural Science Foundation of China

issue (Ennis and Ennis, 2010; Xia and Chen, 2011).

Hypothesis testing is a method for determining the probability of an observed event that occurs only by chance. From the viewpoint of statistics, the result of the reliability experiment should be assured via hypothesis testing. However, it is customary to consider that a result is true after validation by a single experiment. This, in fact, initially assumes a null hypothesis to be true. According to statistical theory, if this null hypothesis is not tested, it is possible to accept a statement that is false. Accordingly, it is vitally important to take hypothesis testing into account (Xia and Chen, 2011).

At present, hypothesis testing relies on a statistic that can be used for testing of standard deviations for parameters, statistical evidence and sample size, and two different discrete populations. Its theoretical bases are fuzzy set theory (Xia and Chen, 2011), Hellinger distance method (Basu *et al.*, 2010), chaos theory (Kula and Apaydin, 2009), regression method (Konishi and Nishiyama, 2009), bootstrap methodology (Martin, 2007), and Bayesian theory (de Santis, 2004).

Hypothesis testing as an arbitration policy finds its position of strength in testing the probability distribution and its numerical characteristic of events, but it does not involve a final verification of the uncertainty of the reliability experiment. This is because the statistic that depicts the uncertainty of the reliability function can hardly be obtained only by the current methods. For this reason, by means of fusing the minimum weighted relative entropy norm and the bootstrap, and with the help of defining the intersection set of the estimated truth-value function and its confidence interval function of the three-parameter Weibull distribution, a new hypothesis testing method is proposed to verify the reliability with the three-parameter Weibull distribution. The case studies of simulation lifetime, helicopter component failure, and ceramic material failure are carried out to prove the viability and effectiveness of the proposed method.

2 Concept of weighted relative entropy norm

2.1 Theoretical and experimental value vectors of reliability

The three-parameter Weibull distribution is given by

$$f(t;\eta,\beta,\tau)=\frac{\beta}{\eta}\left(\frac{t-\tau}{\eta}\right)^{\beta-1}\exp\left(-\left(\frac{t-\tau}{\eta}\right)^{\beta}\right),\quad t\geq\tau>0,\quad \beta>0,\quad \eta>0, \tag{1}$$

where $f(t;\eta,\beta,\tau)$ is the three-parameter Weibull distribution, t is a stochastic variable of the lifetime, (η,β,τ) are the Weibull parameters, η is the scale parameter, β is the shape parameter, and τ is the location parameter.

The reliability function with the three-parameter Weibull distribution can be expressed as

$$R(t;\eta,\beta,\tau)=1-\int_{\tau}^{\infty}f(t;\eta,\beta,\tau)\mathrm{d}t=\exp\left(-\left(\frac{t-\tau}{\eta}\right)^{\beta}\right), \tag{2}$$

where $R(t;\eta,\beta,\tau)$ is the reliability function with the three-parameter Weibull distribution.

For many products in engineering practice, the value of the Weibull parameter is unknown and needs to be found with the help of test evaluation. To this end, a lifetime experiment must be conducted. Assume that the lifetime data of a product are obtained by life tests as follows:

$$\boldsymbol{T}=\{t_i\},\quad t_1\leq t_2\leq\ldots\leq t_n,\quad i=1,2,\ldots,n, \tag{3}$$

where $\boldsymbol{T}$ is the lifetime data vector, t_i is the ith lifetime datum in $\boldsymbol{T}$, i is the sequence number of t_i, and n is the number of the data in $\boldsymbol{T}$.

The lifetime data in $\boldsymbol{T}$ are substituted into Eq. (2), and the theoretical value of the reliability can be calculated by

$$\boldsymbol{R}_0=\{R(t_i;\eta,\beta,\tau)\}, \tag{4}$$

where $R(t_i;\eta,\beta,\tau)$ is the theoretical value of the reliability and $\boldsymbol{R}_0$ is the theoretical value vector of the reliability.

Under the condition of unknown Weibull parameters, Johnson (1970)'s and Nelson (1990)'s methods can be employed to make a nonparametric estimation for the empirical values of the reliability:

$$\boldsymbol{R}_1=\{r(t_i)\}, \tag{5}$$

where $r(t_i)$ is the ith empirical value of the reliability and $\boldsymbol{R}_1$ is the empirical value vector of the reliability.

According to Johnson (1970)'s and Nelson (1990)'s methods, the empirical value of the reliability can be expressed as an expectation (Eq. (6a)) and a median (Eq. (6b)), resepctively:

$$r(t_i) = 1 - \frac{i}{n+1}, \quad i = 1, 2, ..., n, \tag{6a}$$

$$r(t_i) = 1 - \frac{i - 0.3}{n + 0.4}. \tag{6b}$$

In reliability evaluation, the empirical value $r(t_i)$ in $\boldsymbol{R}_1$ can be considered as the experimental value of the reliability due to its corresponding to and depending on the lifetime datum t_i in $\boldsymbol{T}$, and $\boldsymbol{R}_1$ can accordingly be called the experimental value vector of the reliability.

According to Harris (1991) and Jiang and Zuo (1999), the difference between results obtained by Johnson (1970)'s method and Nelson (1990)'s method is very small if n>8. In engineering practice, such as reliability analysis of mechanical products, Nelson (1990)'s method is in common use.

Now, two vectors $\boldsymbol{R}_0$ and $\boldsymbol{R}_1$ are obtained, which are the theoretical and the experimental value vectors, respectively. Under the given criterion, they can be used for the estimation of three Weibull parameters and their confidence intervals by means of the weighted relative entropy norm and bootstrap.

2.2 Definition and property of weighted relative entropy norm

Definition 1 The relative difference information of $\boldsymbol{R}_1$ to $\boldsymbol{R}_0$ is defined as

$$\boldsymbol{L} = \boldsymbol{L}(\boldsymbol{R}_0, \boldsymbol{R}_1) = \left\{ r(t_i) \left| \ln \frac{r(t_i)}{R(t_i; \eta, \beta, \tau)} \right| \right\}, \tag{7}$$

where $r(t_i)$ and $R(t_i;\ \eta,\ \beta,\ \tau)$ are from $\boldsymbol{R}_1$ and $\boldsymbol{R}_0$, respectively.

The relative difference information $\boldsymbol{L}$ is, in fact, the relative entropy vectors of $\boldsymbol{R}_1$ to $\boldsymbol{R}_0$. According to relative entropy theory (Woodbury and Ulrych, 1998; Woodbury, 2004), the smaller is the relative difference information of $\boldsymbol{R}_1$ to $\boldsymbol{R}_0$, the smaller is the norm of $\boldsymbol{L}$.

Definition 2 The weight vector is defined as

$$\boldsymbol{\omega} = \boldsymbol{\omega}(\boldsymbol{R}_0, \boldsymbol{R}_1) = \frac{\{R(t_i; \eta, \beta, \tau) r(t_i)\}}{\| \boldsymbol{R}_0 \|_2 \| \boldsymbol{R}_1 \|_2}. \tag{8}$$

In fact, the weight vector $\boldsymbol{\omega}$ is the cosine of angle between $R(t_i;\ \eta,\ \beta,\ \tau)$ and $r(t_i)$. According to vector theory, the smaller is the difference between the directions of the corresponding elements of $\boldsymbol{R}_0$ and $\boldsymbol{R}_1$, the larger is the norm of $\boldsymbol{\omega}$.

If the equal weight is regarded, $\boldsymbol{\omega}$ becomes a unit vector of size n.

Definition 3 The weighted relative entropy norm is defined as

$$N(\eta, \beta, \tau) = \| \boldsymbol{\omega L} \|_p, \tag{9}$$

where p=1, 2, and ∞, which correspond to 1-norm, 2-norm, and ∞-norm, respectively.

It can be seen from Eqs. (7)–(9) that the smaller is the difference between the directions of the corresponding elements of $\boldsymbol{R}_0$ and $\boldsymbol{R}_1$, the larger is the weight of $\boldsymbol{L}$. Therefore, a smaller weighted relative entropy norm $N(\eta,\ \beta,\ \tau)$ indicates smaller relative difference information of $\boldsymbol{R}_1$ to $\boldsymbol{R}_0$. Two properties of the weighted relative entropy norm can then be inferred below:

Property 1 It is obvious from Definitions 1–3 that the smaller is the value of $N(\eta, \beta, \tau)$, the smaller is the relative difference information of $\boldsymbol{R}_1$ to $\boldsymbol{R}_0$.

Property 2 It is obvious from Definitions 1–3 that $N(\eta, \beta, \tau)$ is approaching zero if $\boldsymbol{R}_1$ is approaching $\boldsymbol{R}_0$.

3 Theorem and proof of minimum weighted relative entropy norm

Theorem 1 Given $\boldsymbol{R}_1$ along with t_i (i=1, 2, …, n), a theoretical function vector $\boldsymbol{R}_0$ along with a continuous variable t is adopted to fit to $\boldsymbol{R}_1$ by $N(\eta, \beta, \tau)$, an optimal estimation (η_{opt}, β_{opt}, τ_{opt}) of (η, β, τ) makes $N(\eta, \beta, \tau)$ the minimum. This is the theorem of the minimum weighted relative entropy norm, which is expressed as

$$N_{\text{opt}}(\eta_{\text{opt}}, \beta_{\text{opt}}, \tau_{\text{opt}}) = \min_{(\eta, \beta, \tau)} \| \boldsymbol{\omega L} \|_p. \tag{10}$$

Proof Let $\boldsymbol{R}_0=\{R(t;\ \eta, \beta, \tau)\}$. When $(\eta, \beta, \tau)=(\eta_{\text{opt}}, \beta_{\text{opt}}, \tau_{\text{opt}})$ and $\{R(t;\ \eta, \beta, \tau)\}=\{R(t;\ \eta_{\text{opt}}, \beta_{\text{opt}}, \tau_{\text{opt}})\}$, $N_{\text{opt}}(\eta_{\text{opt}}, \beta_{\text{opt}}, \tau_{\text{opt}})\leq N(\eta, \beta, \tau)$ for all (η, β, τ). Therefore, $(\eta_{\text{opt}}, \beta_{\text{opt}}, \tau_{\text{opt}})$ satisfying Eq. (10) is an optimal estimation for (η, β, τ).

Theorem 1 means that the estimated value $(\eta_{\text{opt}}, \beta_{\text{opt}}, \tau_{\text{opt}})$ obtained by the minimum weighted relative entropy norm can assure the least relative difference information of $\boldsymbol{R}_1$ to $\boldsymbol{R}_0$. This can lay the foundation for establishing the criterion that is used for an extraction of the optimal information of the Weibull parameters.

4 Criteria of minimum weighted relative entropy norm

With the help of Definition 3 and Theorem 1, six criteria of the minimum weighted relative entropy norm can be established to extract the optimal information of the Weibull parameters.

Criterion 1 This is the minimum 1-norm criterion that is formulated as

$$N_1(\eta_1, \beta_1, \tau_1) = \min_{(\eta, \beta, \tau)} \|\boldsymbol{L}\|_1 . \tag{11}$$

Criterion 1 means that an optimal estimation $(\eta_1, \beta_1, \tau_1)$ for (η, β, τ) makes the 1-norm of the relative difference information of $\boldsymbol{R}_1$ to $\boldsymbol{R}_0$ minimum.

The estimated value $(\eta_1, \beta_1, \tau_1)$ obtained by the minimum 1-norm criterion can assure that the sum of the absolute values of the relative difference information between the corresponding elements of $\boldsymbol{R}_0$ and $\boldsymbol{R}_1$ takes the minimum. This reveals the overall consistency of the information of $\boldsymbol{R}_0$ and $\boldsymbol{R}_1$ in the numerical values on the n-dimensional vector space.

Criterion 2 This is the minimum 2-norm criterion that is formulated as

$$N_2(\eta_2, \beta_2, \tau_2) = \min_{(\eta, \beta, \tau)} \|\boldsymbol{L}\|_2 . \tag{12}$$

Criterion 2 means that an optimal estimation $(\eta_2, \beta_2, \tau_2)$ for (η, β, τ) makes the 2-norm of the relative difference information of $\boldsymbol{R}_1$ to $\boldsymbol{R}_0$ minimum.

The estimated value $(\eta_2, \beta_2, \tau_2)$ obtained by the minimum 2-norm criterion can assure that the sum of the squared difference information between the corresponding elements of $\boldsymbol{R}_0$ and $\boldsymbol{R}_1$ takes the minimum. This reveals the overall consistency of the information of $\boldsymbol{R}_0$ and $\boldsymbol{R}_1$ in the distances on the n-dimensional vector space.

Criterion 3 This is the minimum ∞-norm criterion that is formulated as

$$N_3(\eta_3, \beta_3, \tau_3) = \min_{(\eta, \beta, \tau)} \|\boldsymbol{L}\|_\infty . \tag{13}$$

Criterion 3 means that an optimal estimation $(\eta_3, \beta_3, \tau_3)$ for (η, β, τ) makes the ∞-norm of the relative difference information of $\boldsymbol{R}_1$ to $\boldsymbol{R}_0$ minimum.

The estimated value $(\eta_3, \beta_3, \tau_3)$ obtained by the minimum ∞-norm criterion can assure that the largest relative difference information between the corresponding elements of $\boldsymbol{R}_0$ and $\boldsymbol{R}_1$ takes the minimum. This reveals the consistency of the largest relative difference information of $\boldsymbol{R}_1$ to $\boldsymbol{R}_0$ in the numerical values on the n-dimensional vector space.

Criterion 4 This is the minimum weighted 1-norm criterion that is formulated as

$$N_4(\eta_4, \beta_4, \tau_4) = \min_{(\eta, \beta, \tau)} \|\boldsymbol{\omega L}\|_1 . \tag{14}$$

Criterion 4 means that an optimal estimation $(\eta_4, \beta_4, \tau_4)$ for (η, β, τ) makes the weighted 1-norm of the relative difference information of $\boldsymbol{R}_1$ to $\boldsymbol{R}_0$ minimum.

The estimated value $(\eta_4, \beta_4, \tau_4)$ obtained by the minimum weighted 1-norm criterion can assure that the sum of the weighted absolute values of the relative difference information between the corresponding elements of $\boldsymbol{R}_0$ and $\boldsymbol{R}_1$ takes the minimum. This reveals the overall consistency of the information of $\boldsymbol{R}_0$ and $\boldsymbol{R}_1$ in the directional weighted numerical values on the n-dimensional vector space.

Criterion 5 This is the minimum weighted 2-norm criterion that is formulated as

$$N_5(\eta_5, \beta_5, \tau_5) = \min_{(\eta, \beta, \tau)} \|\boldsymbol{\omega L}\|_2 . \tag{15}$$

Criterion 5 means that an optimal estimation $(\eta_5, \beta_5, \tau_5)$ for (η, β, τ) makes the weighted 2-norm of the relative difference information of $\boldsymbol{R}_1$ to $\boldsymbol{R}_0$ minimum.

The estimated value $(\eta_5, \beta_5, \tau_5)$ obtained by the minimum weighted 2-norm criterion can assure that the sum of the weighted squared difference information between the corresponding elements of $\boldsymbol{R}_0$ and $\boldsymbol{R}_1$

takes the minimum. This reveals the overall consistency of the information of $\boldsymbol{R}_0$ and $\boldsymbol{R}_1$ in the directional weighted distances on the n-dimensional vector space.

Criterion 6 This is the minimum weighted ∞-norm criterion that is formulated as

$$N_6(\eta_6, \beta_6, \tau_6) = \min_{(\eta, \beta, \tau)} \|\boldsymbol{\omega L}\|_\infty . \quad (16)$$

Criterion 6 means that an optimal estimation (η_6, β_6, τ_6) for (η, β, τ) makes the weighted ∞-norm of the relative difference information of $\boldsymbol{R}_1$ to $\boldsymbol{R}_0$ minimum.

The estimated value (η_6, β_6, τ_6) obtained by the minimum weighted ∞-norm criterion can assure that the largest weighted relative difference information between the corresponding elements of $\boldsymbol{R}_0$ and $\boldsymbol{R}_1$ takes the minimum. This reveals the consistency of the largest relative difference information of $\boldsymbol{R}_1$ to $\boldsymbol{R}_0$ in the directional weighted numerical values on the n-dimensional vector space.

Multiple criteria are used because different results reveal different natures of the studied product reliability (Xia, 2012). That is to say, the larger is the number of the criteria, the more useful is the information. This is beneficial in subsequently establishing the probability density function of the Weibull parameter. In addition, only the aforementioned six norms can be employed at present owing to the limited number of mathematical methods (Feng *et al.*, 2012). Therefore, the six norm criteria are selected. According to norm theory, each of these results can be regarded as one datum that is an estimate for the real value of the Weibull parameters (η, β, τ). It follows that the optimal information of the Weibull parameters (η, β, τ) is extracted.

5 Optimal information vector of the Weibull parameters

To facilitate the description, η, β, and τ are noted by θ.

Using the six criteria in Section 4, the optimal information vector of the Weibull parameter θ can be obtained as

$$\boldsymbol{\theta} = \{\theta_m\}, \quad m = 1,\ 2,\ ...,\ 6, \quad (17)$$

where $\boldsymbol{\theta}$ stands for the optimal information vector of θ, θ_m is the mth estimated value solved by the mth criterion, and m is the mth criterion.

According to Bayesian statistics (Lee and Pan, 2012; de Santis, 2004), the Weibull parameter θ can be regarded as a stochastic variable with a probability distribution. As previously mentioned, the data in $\boldsymbol{\theta}$ are the optimal estimate for θ and reflect the useful information of different sides of θ. Thus, the probability distribution of the Weibull parameter θ can be evaluated by $\boldsymbol{\theta}$.

6 Assessment of three-parameter Weibull distribution

The estimated truth-value function and its confidence interval function of the three-parameter Weibull distribution must be established to conduct the hypothesis testing for reliability. This needs to estimate the probability density function, the estimated truth-value, and the confidence interval of θ.

6.1 Estimation of the probability density function of Weibull parameters

According to the classical statistical theory, large amounts of data must be obtained to estimate the probability density function of θ. Unfortunately, the amount of the parameter data in $\boldsymbol{\theta}$ is too small due to the limited number of criteria in norm theory. To address the problem, the bootstrap (Efron, 1979; Xia, 2012) can be employed. The bootstrap is one of the prevalent methods for generation of many data and imitation of an unknown probability distribution even with little data. According to Xia (2012), the steps to apply the bootstrap are as follows:

1. Let constant B=100 000, and variable b=1, where B is the number of the resampling samples and b is the bth equiprobable resampling.

2. Let one datum be drawn by an equiprobable resampling with replacement from $\boldsymbol{\theta}$.

3. Repeat step 2 six times, so that six data can be sampled.

4. Calculate the mean $\theta(b)$ of the six data, which is considered as one of the data in the generated information vector $\boldsymbol{\Theta}$ of θ.

5. Add 1 to b.

6. If b>B, go to step 7; otherwise, go to step 2.

7. Let the generated information vector be of size B=100 000, so that many generated parameter data are obtained.

According to the bootstrap, an equiprobable resampling with replacement from $\boldsymbol{\theta}$ is implemented, and a large amount of the generated parameter data can then be gained as follows:

$$\boldsymbol{\Theta}=\{\theta(b)\}=\left\{\frac{1}{6}\sum_{m=1}^{6}\theta_b(m)\right\},\quad b=1,2,...,B, \tag{18}$$

where b is the bth equiprobable resampling with replacement from $\boldsymbol{\theta}$, $\theta_b(m)$ is the mth datum obtained in the bth resampling, and $\theta(b)$ is the mean of the six data obtained in the bth resampling.

From Eq. (18), using the histogram method in statistical theory, the probability density function of θ is obtained as

$$\varphi=\varphi(\theta). \tag{19}$$

6.2 Estimated truth-value and its confidence interval of Weibull parameters

The estimated truth-value of θ is defined as

$$\theta_0=\int_{\theta_{\min}}^{\theta_{\max}}\theta\varphi(\theta)\mathrm{d}\theta, \tag{20}$$

where θ_0 is the estimated truth-value of θ, viz., the estimated truth-value (η_0, β_0, τ_0) of (η, β, τ), and $\theta_{\min}$ and $\theta_{\max}$ are the lower limit and the upper limit of the integral interval of θ, respectively.

Letting the significance level $\alpha\in[0, 1]$, the confidence level P is given by

$$P=(1-\alpha)\times 100\%. \tag{21}$$

In the hypothesis testing for reliability, the proposed significance level is 0.1.

Under the condition of the confidence level P, the lower bound θ_L and the upper bound θ_U of θ can be worked out by the integral of $\varphi(\theta)$. The integral equations are as follows:

$$\frac{\alpha}{2}=\int_{\theta_{\min}}^{\theta_L}\varphi(\theta)\mathrm{d}\theta, \tag{22}$$

$$1-\frac{\alpha}{2}=\int_{\theta_{\min}}^{\theta_U}\varphi(\theta)\mathrm{d}\theta. \tag{23}$$

The estimated truth-value (η_0, β_0, τ_0) of (η, β, τ) can be obtained from Eq. (20) and the confidence interval ([η_L, η_U], [β_L, β_U], [τ_L, τ_U]) of (η_0, β_0, τ_0) can be obtained from Eqs. (22) and (23).

6.3 Estimated truth-value function and confidence interval function of three-parameter Weibull distribution

Letting (η, β, τ)=(η_0, β_0, τ_0), then the estimated truth-value function $f_0(t)$ of the three-parameter Weibull distribution is gained from Eq. (1) as follows:

$$f_0(t)=\frac{\beta_0}{\eta_0}\left(\frac{t-\tau_0}{\eta_0}\right)^{\beta_0-1}\exp\left(-\left(\frac{t-\tau_0}{\eta_0}\right)^{\beta_0}\right),\quad t\geq\tau_0>0,\ \beta_0>0,\ \eta_0>0. \tag{24}$$

The confidence interval function of the three-parameter Weibull distribution includes the lower bound function $f_L(t)$ and the upper bound function $f_U(t)$, which are respectively defined as

$$f_L(t)=\min f_j(t),\ f_j(t)\leq f_0(t),\ j=1,2,...,8, \tag{25}$$

$$f_U(t)=\max f_j(t),\ f_j(t)\geq f_0(t),\ j=1,2,...,8, \tag{26}$$

where

$$f_1(t)=\frac{\beta_L}{\eta_L}\left(\frac{t-\tau_L}{\eta_L}\right)^{\beta_L-1}\exp\left(-\left(\frac{t-\tau_L}{\eta_L}\right)^{\beta_L}\right),\quad t\geq\tau_L\geq 0,\ \beta_L>0,\ \eta_L>0, \tag{27}$$

$$f_2(t)=\frac{\beta_L}{\eta_L}\left(\frac{t-\tau_U}{\eta_L}\right)^{\beta_L-1}\exp\left(-\left(\frac{t-\tau_U}{\eta_L}\right)^{\beta_L}\right),\quad t\geq\tau_U\geq\tau_L\geq 0, \tag{28}$$

$$f_3(t)=\frac{\beta_U}{\eta_L}\left(\frac{t-\tau_L}{\eta_L}\right)^{\beta_U-1}\exp\left(-\left(\frac{t-\tau_L}{\eta_L}\right)^{\beta_U}\right),\quad t\geq\tau_L\geq 0,\ \beta_U\geq\beta_L, \tag{29}$$

$$f_4(t)=\frac{\beta_U}{\eta_L}\left(\frac{t-\tau_U}{\eta_L}\right)^{\beta_U-1}\exp\left(-\left(\frac{t-\tau_U}{\eta_L}\right)^{\beta_U}\right),\quad t\geq\tau_U\geq\tau_L\geq 0, \tag{30}$$

$$f_5(t)=\frac{\beta_{\mathrm{L}}}{\eta_{\mathrm{U}}}\left(\frac{t-\tau_{\mathrm{L}}}{\eta_{\mathrm{U}}}\right)^{\beta_{\mathrm{L}}-1}\exp\left(-\left(\frac{t-\tau_{\mathrm{L}}}{\eta_{\mathrm{U}}}\right)^{\beta_{\mathrm{L}}}\right),\quad \eta_{\mathrm{U}}\ge\eta_{\mathrm{L}},\tag{31}$$

$$f_6(t)=\frac{\beta_{\mathrm{L}}}{\eta_{\mathrm{U}}}\left(\frac{t-\tau_{\mathrm{U}}}{\eta_{\mathrm{U}}}\right)^{\beta_{\mathrm{L}}-1}\exp\left(-\left(\frac{t-\tau_{\mathrm{U}}}{\eta_{\mathrm{U}}}\right)^{\beta_{\mathrm{L}}}\right),\quad t\ge\tau_{\mathrm{U}}\ge\tau_{\mathrm{L}}\ge0,\tag{32}$$

$$f_7(t)=\frac{\beta_{\mathrm{U}}}{\eta_{\mathrm{U}}}\left(\frac{t-\tau_{\mathrm{L}}}{\eta_{\mathrm{U}}}\right)^{\beta_{\mathrm{U}}-1}\exp\left(-\left(\frac{t-\tau_{\mathrm{L}}}{\eta_{\mathrm{U}}}\right)^{\beta_{\mathrm{U}}}\right),\quad t\ge\tau_{\mathrm{L}}\ge0,\tag{33}$$

$$f_8(t)=\frac{\beta_{\mathrm{U}}}{\eta_{\mathrm{U}}}\left(\frac{t-\tau_{\mathrm{U}}}{\eta_{\mathrm{U}}}\right)^{\beta_{\mathrm{U}}-1}\exp\left(-\left(\frac{t-\tau_{\mathrm{U}}}{\eta_{\mathrm{U}}}\right)^{\beta_{\mathrm{U}}}\right),\quad t\ge\tau_{\mathrm{U}}\ge\tau_{\mathrm{L}}\ge0.\tag{34}$$

Consider that $f_0(t)$, $f_{\mathrm{L}}(t)$, and $f_{\mathrm{U}}(t)$ as three sets of the continuous variable t. Then hypothesis testing for reliability with the three-parameter Weibull distribution can easily be implemented.

7 Hypothesis testing for reliability with three-parameter Weibull distribution

Definition 4 Define the intersection set $A_{\mathrm{L}}(t)$ of $f_0(t)$ and $f_{\mathrm{L}}(t)$ as

$$A_{\mathrm{L}}(t)=f_0(t)\cap f_{\mathrm{L}}(t),\ t\in\Omega,\tag{35}$$

where Ω is the feasible region of t.

According to Definition 4, the abscissa value $t_{0\mathrm{L}}$ of the intersection point of $f_0(t)$ and $f_{\mathrm{L}}(t)$ as two curves satisfies

$$f_0(t_{0\mathrm{L}})=f_{\mathrm{L}}(t_{0\mathrm{L}}).\tag{36}$$

The area a_{L} of A_{L} is marked by

$$a_{\mathrm{L}}=\mathrm{area}\left(A_{\mathrm{L}}(t)\right),\quad a_{\mathrm{L}}\in[0,1].\tag{37}$$

The area a_{L} is a probability, at which $f_{\mathrm{L}}(t)$ is approaching $f_0(t)$.

Definition 5 Define the intersection set $A_{\mathrm{U}}(t)$ of $f_0(t)$ and $f_{\mathrm{U}}(t)$ as

$$A_{\mathrm{U}}(t)=f_0(t)\cap f_{\mathrm{U}}(t),\quad t\in\Xi,\tag{38}$$

where Ξ is the feasible region of t.

According to Definition 5, the abscissa value $t_{0\mathrm{U}}$ of the intersection point of $f_0(t)$ and $f_{\mathrm{U}}(t)$ as two curves satisfies

$$f_0(t_{0\mathrm{U}})=f_{\mathrm{U}}(t_{0\mathrm{U}}).\tag{39}$$

The area a_{U} of A_{U} is marked by

$$a_{\mathrm{U}}=\mathrm{area}\left(A_{\mathrm{U}}(t)\right),\quad a_{\mathrm{U}}\in[0,1].\tag{40}$$

The area a_{U} is a probability, at which $f_{\mathrm{U}}(t)$ is approaching $f_0(t)$.

Property 3 From the point of view of reliability theory, because the value and trend of $\boldsymbol{R}_0$ are characterized by $f_0(t)$ and the uncertainty and trend of $\boldsymbol{R}_1$ are characterized by $f_{\mathrm{L}}(t)$ and $f_{\mathrm{U}}(t)$, Definitions 4 and 5 show that the mean of the two areas a_{L} and a_{U} ($0.5(a_{\mathrm{L}}+a_{\mathrm{U}})$) is a probability at which $\boldsymbol{R}_1$ is approaching $\boldsymbol{R}_0$. When the mean of a_{L} and a_{U} takes a value 1, the probability of $\boldsymbol{R}_1$ approaching $\boldsymbol{R}_0$ is 100%, which reveals that $\boldsymbol{R}_1$ is just the same as $\boldsymbol{R}_0$. When the mean of a_{L} and a_{U} takes a value 0, the probability of $\boldsymbol{R}_1$ approaching $\boldsymbol{R}_0$ is 0%, which reveals that $\boldsymbol{R}_1$ is totally different from $\boldsymbol{R}_0$.

Definition 6 Define the null hypothesis H_0 and the alternative hypothesis H_1 as follows:

$$H_0:\ \boldsymbol{R}_1=\boldsymbol{R}_0,\tag{41}$$

$$H_1:\ \boldsymbol{R}_1\neq\boldsymbol{R}_0.\tag{42}$$

The null hypothesis H_0 indicates that the relative difference information of $\boldsymbol{R}_1$ to $\boldsymbol{R}_0$ is very small and $\boldsymbol{R}_1$ does not significantly deviate from $\boldsymbol{R}_0$.

Theorem 2 Given a significance level α, the rejection region for the null hypothesis H_0 is

$$0.5(a_{\mathrm{L}}+a_{\mathrm{U}})\le 1-0.5\alpha.\tag{43}$$

The rejection region is along with the confidence level P.

Proof This is a two-sided test according to Definition 6. The probability that the event $\boldsymbol{R}_1=\boldsymbol{R}_0$ occurs is $0.5(a_{\mathrm{L}}+a_{\mathrm{U}})$ according to Definitions 4 and 5 and Property 3. For a small probability event, the probability that the event $\boldsymbol{R}_1=\boldsymbol{R}_0$ cannot occur is $1-0.5(a_{\mathrm{L}}+a_{\mathrm{U}})$ according to the principle of the small probability event. The condition of the small probability event can hence be given by

$$1-0.5(a_L+a_U)\geq k,$$

where k is a very small real number. Consequently, $0.5(a_L+a_U)\leq 1-k$. Considering α as the threshold value of the probability at which the small probability event may occur (viz., $k=0.5\alpha$ for the two-sided test), the inequality $0.5(a_L+a_U)\leq 1-0.5\alpha$ is attested.

In terms of Definition 6 and Theorem 2, if the mean of a_L and a_U satisfies the rejection region, H_0 is rejected, indicating that at the confidence level P, the relative difference information of $\boldsymbol{R}_1$ to $\boldsymbol{R}_0$ is significant; or else, H_0 is not rejected, indicating that at the confidence level P, the relative difference information of $\boldsymbol{R}_1$ to $\boldsymbol{R}_0$ is not significant.

8 Procedures of the hypothesis testing method

The procedures of the hypothesis testing method proposed are summarized as follows:

1. Conduct a lifetime experiment and obtain the lifetime data vector.
2. Find the optimal information vector of the Weibull parameter by six criteria based on the lifetime data vector.
3. Obtain the generated information vector of the Weibull parameters with the bootstrap based on the optimal information vector.
4. Establish the probability density function of the Weibull parameters via the histogram method based on the generated information vector.
5. Evaluate the estimated truth-value and its confidence interval of Weibull parameters.
6. Structure the estimated truth-value function and its confidence interval function of the three-parameter Weibull distribution.
7. Conduct hypothesis testing for the reliability with the three-parameter Weibull distribution based on the rejection region relied on the area of the intersection set.

9 Case study and discussion

9.1 Simulation case

Let $(\eta, \beta, \tau)=(30, 2.5, 10)$ be the truth-value. Then the three-parameter Weibull distribution is simulated by a computer system and the lifetime data (d) are obtained as (n=9)

$$\boldsymbol{T}=(22.1953, 26.4647, 29.8623, 32.9314, 35.9090, 38.9691, 42.3123, 46.2903, 51.8801).$$

In this case, the truth is that no difference information exists between the theoretical value vector $\boldsymbol{R}_0$ and experimental value vector $\boldsymbol{R}_1$. The aim of this case is to prove that this is true, with the help of the hypothesis testing method proposed.

Test 1: H_0: $\boldsymbol{R}_1=\boldsymbol{R}_0$ vs. H_1: $\boldsymbol{R}_1\neq\boldsymbol{R}_0$.

Let $\alpha=0.1$, and then $P=90\%$. The results obtained using the method proposed are shown in Table 1 and Figs. 1 and 2.

Table 1 Estimated truth-value of Weibull parameters in a simulation case

Weibull parameter	Estimated truth-value	Truth-value	Relative error (%)
η	30.1387	30	0.46
β	2.5218	2.5	0.87
τ	9.8418	10	1.58

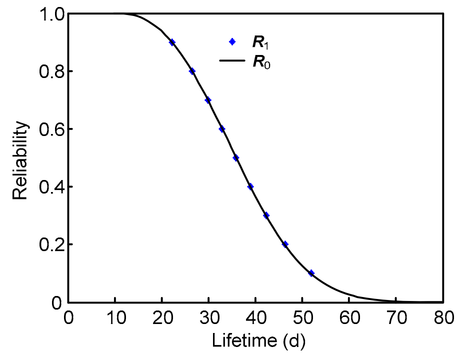

Fig. 1 Theoretical and experimental value vectors of reliability in a simulation case

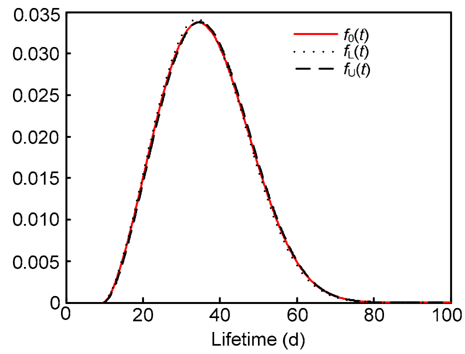

Fig. 2 Estimated truth-value function $f_0(t)$, lower bound function $f_L(t)$, and upper bound function $f_U(t)$ in a simulation case

Table 1 presents the estimated truth-value (η_0, β_0, τ_0) of the Weibull parameters (η, β, τ). It can be seen that the relative errors of the estimated results of the three Weibull parameters are very small. This indicates that the method proposed is effective and reliable.

Fig. 1 shows the results of $\boldsymbol{R}_0$ and $\boldsymbol{R}_1$. Obviously, $\boldsymbol{R}_1$ is almost coincident with $\boldsymbol{R}_0$.

Fig. 2 shows the results of the estimated truth-value function $f_0(t)$, the lower bound function $f_L(t)$, and the upper bound function $f_U(t)$. The abscissa value of the intersection point of $f_0(t)$ and $f_L(t)$ is t_{0L}=40.0718, and the area of the intersection set $A_L(t)$ of $f_0(t)$ and $f_L(t)$ is a_L=0.9887. The abscissa value of the intersection point of $f_0(t)$ and $f_U(t)$ is t_{0U}=33.6633, and the area of the intersection set $A_U(t)$ of $f_0(t)$ and $f_U(t)$ is a_U=0.9914.

It is easy to see that $0.5(a_L+a_U)=0.99>1-0.5\alpha=0.95$; that is, the mean of a_L and a_U does not satisfy the rejection region. The hypothesis H_0 can accordingly be accepted. The conclusion is that at the 90% confidence level, the relative difference information of $\boldsymbol{R}_1$ to $\boldsymbol{R}_0$ is not significant. This is true as stated in the beginning of this case.

In this case, if α=0.05 is considered, the related results are as follows:

$$P=95\%,\ a_L=0.9876,\ a_U=0.9901,$$
$$0.5(a_L+a_U)=0.989>1-0.5\alpha=0.975.$$

The mean of a_L and a_U does not satisfy the rejection region. The hypothesis H_0 can accordingly be accepted. The conclusion is that at the 95% confidence level, the relative difference information of $\boldsymbol{R}_1$ to $\boldsymbol{R}_0$ is not significant. The conclusion is the same as above (α=0.1), although the significance is increased. This is true as stated in the beginning of this case.

To test the robustness of the proposed method, a Gaussian noise, $N(0, 0.2^2)$, viz., with mean of 0 and standard deviation of 0.2, is added to $\boldsymbol{T}$. The results of the estimated truth-value are presented in Table 2. It can be seen that the relative errors of the estimated results of the three Weibull parameters are still very small, showing that the proposed method is robust.

In this case by adding the noise term, if α=0.05 is considered, the related results are as follows:

$$P=95\%,\ a_L=0.9877,\ a_U=0.9906,$$
$$0.5(a_L+a_U)=0.989>1-0.5\alpha=0.975.$$

Table 2 Estimated truth-value of Weibull parameters in a simulation case with Gaussian noise

Weibull parameter	Estimated truth-value	Truth-value	Relative error (%)
η	30.41634	30	1.39
β	2.54684	2.5	1.87
τ	9.60825	10	3.92

The mean of a_L and a_U does not satisfy the rejection region. The hypothesis H_0 can accordingly be accepted. The conclusion is that at the 95% confidence level, the relative difference information of $\boldsymbol{R}_1$ to $\boldsymbol{R}_0$ is not significant. The conclusion is the same as above (α=0.1), although the significance is increased and the noise term is added. This is true as stated in the beginning of this case.

9.2 Helicopter component case

In failure analysis, the lifetime data (h) of a helicopter component are obtained as follows (n=13) (Luxhoj and Shyur, 1995):

$$\boldsymbol{T}=(156.5, 213.4, 265.7, 265.7, 337.7, 337.7, 406.3, 573.5, 573.5, 644.6, 744.8, 774.8, 1023.6).$$

The aim in this case is to test whether the lifetime data deviate from the three-parameter Weibull distribution.

Test 2: H_0: $\boldsymbol{R}_1=\boldsymbol{R}_0$ vs. H_1: $\boldsymbol{R}_1\neq\boldsymbol{R}_0$.

Let α=0.1, and then P=90%. The results obtained using the method proposed are shown in Figs. 3 and 4.

Fig. 3 shows the results of $\boldsymbol{R}_0$ and $\boldsymbol{R}_1$. Obviously, although $\boldsymbol{R}_1$ has a similar trend to $\boldsymbol{R}_0$, its uncertainty and fluctuation relative to $\boldsymbol{R}_0$ are large.

Fig. 4 shows the results of the estimated truth-value function $f_0(t)$, the lower bound function $f_L(t)$, and the upper bound function $f_U(t)$. The abscissa value of the intersection point of $f_0(t)$ and $f_L(t)$ is t_{0L}=449.162, and the area of the intersection set $A_L(t)$ of $f_0(t)$ and $f_L(t)$ is a_L=0.9336. The abscissa value of the intersection point of $f_0(t)$ and $f_U(t)$ is t_{0U}=372.4257, and the area of the intersection set $A_U(t)$ of $f_0(t)$ and $f_U(t)$ is a_U=0.9475.

It is easy to see that $0.5(a_L+a_U)=0.94<1-0.5\alpha=0.95$; that is, the mean of a_L and a_U satisfies the rejection region. The hypothesis H_0 can accordingly be rejected. The conclusion is that at the 90% confidence level, the relative difference information of $\boldsymbol{R}_1$ to $\boldsymbol{R}_0$ is significant and the lifetime data tested deviate from the three-parameter Weibull distribution.

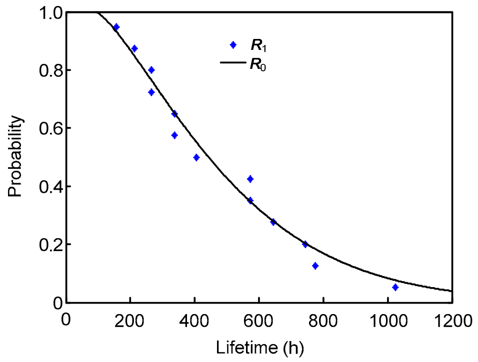

Fig. 3 Theoretical and experimental value vectors of reliability in a helicopter component case

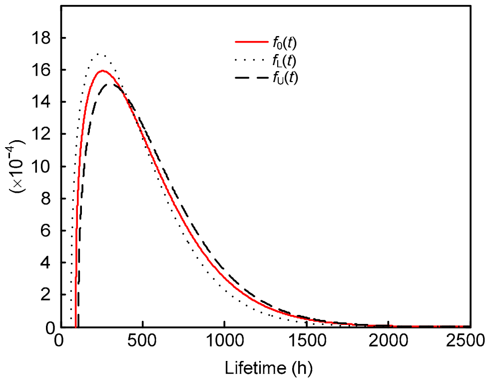

Fig. 4 Estimated truth-value function $f_0(t)$, lower bound function $f_L(t)$, and upper bound function $f_U(t)$ in a helicopter component case

9.3 Ceramic material case

In life tests, Duffy *et al*. (1993) obtained the lifetime data (h) of aluminum oxide ceramics, as follows (n=35):

$\boldsymbol{T}$=(307, 308, 322, 328, 328, 329, 331, 332, 335, 337, 343, 345, 347, 350, 352, 353, 355, 356, 357, 364, 371, 373, 374, 375, 376, 376, 381, 385, 388, 395, 402, 411, 413, 415, 456).

Suppose the failure lifetime of aluminum oxide ceramics is of the three-parameter Weibull distribution. The aim of this case is to test whether the result of the lifetime experiment is credible.

Test 3: H_0: $\boldsymbol{R}_1=\boldsymbol{R}_0$ vs. H_1: $\boldsymbol{R}_1\neq\boldsymbol{R}_0$.

Let α=0.1, and then P=90%. The results obtained using the method proposed are shown in Figs. 5 and 6.

Fig. 5 shows the results of $\boldsymbol{R}_0$ and $\boldsymbol{R}_1$. Obviously, $\boldsymbol{R}_1$ has a similar trend to $\boldsymbol{R}_0$ and its uncertainty and fluctuation relative to $\boldsymbol{R}_0$ are also very small.

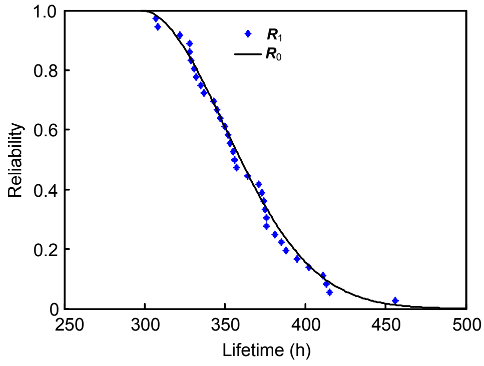

Fig. 5 Theoretical and experimental value vectors of reliability in a ceramic material case

Fig. 6 shows the results of the estimated truth-value function $f_0(t)$, the lower bound function $f_L(t)$, and the upper bound function $f_U(t)$. The abscissa value of the intersection point of $f_0(t)$ and $f_L(t)$ is t_{0L}=376.520, and the area of the intersection set $A_L(t)$ of $f_0(t)$ and $f_L(t)$ is a_L=0.9534. The abscissa value of the intersection point of $f_0(t)$ and $f_U(t)$ is t_{0U}=350.138, and the area of the intersection set $A_U(t)$ of $f_0(t)$ and $f_U(t)$ is a_U=0.9601.

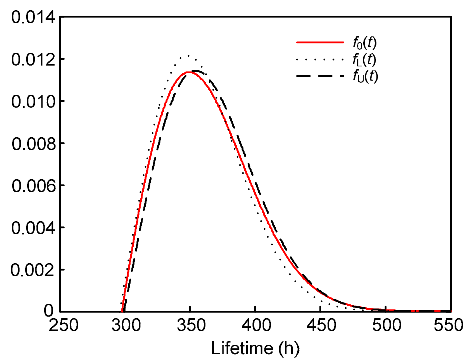

Fig. 6 Estimated truth-value function $f_0(t)$, lower bound function $f_L(t)$, and upper bound function $f_U(t)$ in a ceramic material case

It is easy to see that $0.5(a_L+a_U)=0.957>1-0.5\alpha=0.95$; that is, the mean of a_L and a_U does not satisfy the rejection region. The hypothesis H_0 can accordingly be accepted. The conclusion is that at the 90% confidence level, the relative difference information of $\boldsymbol{R}_1$ to $\boldsymbol{R}_0$ is not significant and the result of the lifetime experiment tested is credible.

In this case, if α=0.05 is considered, the related results are as follows:

$$P=95\%,\ a_L=0.9336,\ a_U=0.9475,$$
$$0.5(a_L+a_U)=0.941<1-0.5\alpha=0.975.$$

The mean of a_L and a_U satisfies the rejection region. The hypothesis H_0 can accordingly be rejected. The conclusion is that at the 95% confidence level, the relative difference information of $\boldsymbol{R}_1$ to $\boldsymbol{R}_0$ is significant and the result of the lifetime experiment tested is not credible. The conclusion is not the same as above (α=0.1), as the significance is increased. As shown in Fig. 5, although the uncertainty and fluctuation of $\boldsymbol{R}_1$ relative to $\boldsymbol{R}_0$ are very small, it is still identified due to higher significance.

From the three cases above, the method proposed is good at testing of hypothesis for the reliability with the three-parameter Weibull distribution as its results reflect the practical situation of lifetime experiments.

The mechanism of the proposed method lies in the discovery of the optimal information vector of the Weibull parameters by six criteria based on the relative difference information of the experimental value vector to the theoretical function vector. On this ground, the estimated truth-value function and its confidence interval function of the three-parameter Weibull distribution are established with the bootstrap, and the rejection region used in hypothesis testing is demarcated via the area of the intersection set of the estimated truth-value function and its confidence interval function of the three-parameter Weibull distribution.

10 Conclusions

The proposed criteria of the minimum weighted relative entropy norm are based on the relative difference information of the experimental value vector to the theoretical value vector of the reliability, and can be used for extraction of the optimal information vector of the parameters of the three-parameter Weibull distribution in the reliability experiment of a product lifetime.

The proposed rejection region is able to make the hypothesis testing for the reliability experiment via the area of the intersection set of the estimated truth-value function and its confidence interval function of the three-parameter Weibull distribution.

The case studies of simulation lifetime, helicopter component failure, and ceramic material failure indicate that the proposed hypothesis testing method reflect the practical situation of the reliability experiment.

References

Abbasi, B., Niaki, S.T.A., Khalife, M.A., Faize, Y., 2011. A hybrid variable neighborhood search and simulated annealing algorithm to estimate the three parameters of the Weibull distribution. *Exp. Syst. Appl.*, **38**(1):700-708. [doi:10.1016/j.eswa.2010.07.022]

Bartkute-Norkuniene, V., Sakalauskas, L., 2011. Consistent estimator of the shape parameter of three-dimensional Weibull distribution. *Commun. Stat. Theory Meth.*, **40**(16): 2985-2996. [doi:10.1080/03610926.2011.562785]

Basu, A., Manda, A., Pardo, L., 2010. Hypothesis testing for two discrete populations based on the Hellinger distance. *Stat. Prob. Lett.*, **80**(3-4):206-214. [doi:10.1016/j.spl.2009.10.008]

Chatterjee, S., Bandopadhyay, S., 2012. Reliability estimation using a genetic algorithm-based artificial neural network: an application to a load-haul-dump machine. *Exp. Syst. Appl.*, **39**(12):10943-10951. [doi:10.1016/j.eswa.2012.03.030]

Deng, J., Gu, D.S., Li, X.B., 2004. Parameters and quantile estimation for fatigue life distribution using probability weighted moments. *Chin. J. Comput. Mech.*, **21**(5): 609-613 (in Chinese).

de Santis, F., 2004. Statistical evidence and sample size determination for Bayesian hypothesis testing. *J. Stat. Plan. Infer.*, **124**(1):121-144. [doi:10.1016/S0378-3758(03)00198-8]

Duffy, S.F., Palko, J.L., Gyekenyesi, J.P., 1993. Structural reliability analysis of laminated CMC components. *J. Eng. Gas Turb. Power*, **115**(1):103-108. [doi:10.1115/1.2906663]

Efron, B., 1979. Bootstrap methods. *Ann. Stat.*, **7**(1):1-36. [doi:10.1214/aos/1176344552]

El-Adll, M.E., 2011. Predicting future lifetime based on random number of three parameters Weibull distribution. *Math. Comput. Simul.*, **81**(9):1842-1854. [doi:10.1016/j.matcom.2011.02.003]

Ennis, D.M., Ennis, J.M., 2010. Equivalence hypothesis testing. *Food Qual. Pref.* **21**(3):253-256. [doi:10.1016/j.foodqual.2009.06.005]

Feng, D.C., Chen, F., Xu, W.L., 2012. Learning robust principal components from L1-norm maximization. *J. Zhejiang Univ.-Sci. C (Comput. & Electron.)*, **13**(12):901-908. [doi:10.1631/jzus.C1200180]

Harris, T.A., 1991. Rolling Bearing Analysis (3rd Ed.). John Wiley & Sons, New York.

Jiang, R.Y., Zuo, M.J., 1999. Reliability Model and Application. China Machine Press, Beijing (in Chinese).

Johnson, L.G., 1970. Theory and Technique of Variation Research. Elsevier, New York.

Kaplan, E.L., Meier, P., 1958. Nonparametric estimation from incomplete observations. *J. Am. Stat. Assoc.*, **53**(282): 457-481. [doi:10.1080/01621459.1958.10501452]

Konishi, Y., Nishiyama, Y., 2009. Hypothesis testing in rank-size rule regression. *Math. Comput. Simul.*, **79**(9): 2869-2878. [doi:10.1016/j.matcom.2008.10.012]

Kula, K.S., Apaydin, A., 2009. Hypotheses testing for fuzzy robust regression parameters. *Chaos Solit. Fract.*, **42**(4): 2129-2134. [doi:10.1016/j.chaos.2009.03.140]

Lee, J., Pan, R., 2012. Bayesian analysis of step-stress accelerated life test with exponential distribution. *Qual. Rel. Eng. Int.*, **28**(3):353-361. [doi:10.1002/qre.1251]

Li, Q., Liu, S.L., Pan, X.H., Zheng, S.Y., 2012. A new method for studying the 3D transient flow of misaligned journal bearings in flexible rotor-bearing systems. *J. Zhejiang Univ.-Sci. A (Appl. Phys. & Eng.)*, **13**(4):293-310. [doi:10.1631/jzus.A1100228]

Luxhoj, T.J., Shyur, H.J., 1995. Reliability curve fitting for aging helicopter components. *Rel. Eng. Syst. Safety*, **48**(3): 229-234. [doi:10.1016/0951-8320(95)00018-W]

Martin, M.A., 2007. Bootstrap hypothesis testing for some common statistical problems: a critical evaluation of size and power properties. *Comput. Stat. Data Anal.*, **51**(12): 6321-6342. [doi:10.1016/j.csda.2007.01.020]

Nelson, W., 1990. Accelerated Testing: Statistical Models, Test Plans and Data Analysis. John Wiley & Sons, New York.

Niesłony, A., Ruzicka, M., Papuga, J., Hodr, A., Balda, M., Svoboda, J., 2012. Fatigue life prediction for broad-band multiaxial loading with various PSD curve shapes. *Int. J. Fatigue*, **44**:74-88. [doi:10.1016/j.ijfatigue.2012.05.014]

Pierce, D.M., Zeyen, B., Huigens, B.M., Fitzgerald, A.M., 2011. Predicting the failure probability of device features in MEMS. *IEEE Trans. Dev. Mat. Rel.*, **11**(3):433-441. [doi:10.1109/TDMR.2011.2159117]

Prawoto, Y., Dillon, B., 2012. Failure analysis and life assessment of coating: the use of mixed mode stress intensity factors in coating and other surface engineering life assessment. *J. Fail. Anal. Prev.*, **12**(2):190-197. [doi:10.1007/s11668-011-9525-1]

Qian, L.F., 2012. The Fisher information matrix for a three-parameter exponentiated Weibull distribution under type II censoring. *Stat. Methodol.*, **9**(3):320-329. [doi:10.1016/j.stamet.2011.08.007]

Shafieezadeh, A., Ellingwood, B.R., 2012. Confidence intervals for reliability indices using likelihood ratio statistics. *Struct. Safety*, **38**:48-55. [doi:10.1016/j.strusafe.2012.04.002]

Tan, P., He, W.T., Lin, J., Zhao, H.M., Chu, J., 2011. Design and reliability, availability, maintainability, and safety analysis of a high availability quadruple vital computer system. *J. Zhejiang Univ.-Sci. A (Appl. Phys. & Eng.)*, **12**(12):926-935. [doi:10.1631/jzus.A11GT003]

Toasa Caiza, P.D., Ummenhofer, T., 2011. General probability weighted moments for the three-parameter Weibull distribution and their application in *S-N* curves modeling. *Int. J. Fatigue*, **33**(12):1533-1538. [doi:10.1016/j.ijfatigue.2011.06.009]

Woodbury, A.D., 2004. A FORTRAN program to produce minimum relative entropy distributions. *Comput. Geosci.*, **30**(1):131-138. [doi:10.1016/j.cageo.2003.09.001]

Woodbury, A.D., Ulrych, T.J., 1998. Minimum relative entropy and probabilistic inversion in groundwater hydrology. *Stoch. Hydrol. Hydraul.*, **12**(5):317-358. [doi:10.1007/s004770050024]

Xia, X.T., 2012. Forecasting method for product reliability along with performance data. *J. Fail. Anal. Prev.*, **12**(5): 532-540. [doi:10.1007/s11668-012-9592-y]

Xia, X.T., Chen, J.F., 2011. Fuzzy hypothesis testing and time series analysis of rolling bearing quality. *J. Test. Eval.*, **39**(6):1144-1151. [doi:10.1520/JTE103371]

Zhang, Y., Zheng, Y.M., Yang, M., Li, H., Jin, Z.H., 2012. Design and implementation of the highly-reliable, low-cost housekeeping system in the ZDPS-1A pico-satellite. *J. Zhejiang Univ.-Sci. C (Comput. & Electron.)*, **13**(2): 83-89. [doi:10.1631/jzus.C1100079]

Zhao, X.P., Lu, Z.Z., Wang, W.H., 2010. A practical and effective method for estimating confidence limits of population percentile and reliability of three-parameter Weibull distribution on probability weighted moment. *J. Northw. Polytechn. Univ.*, **38**(3):470-475 (in Chinese).