Journal of Zhejiang University-SCIENCE C (Computers & Electronics)
ISSN 1869-1951 (Print); ISSN 1869-196X (Online)
www.zju.edu.cn/jzus; www.springerlink.com
E-mail: jzus@zju.edu.cn

Exponential stability of nonlinear impulsive switched systems with stable and unstable subsystems*

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Received May 9, 2013; Revision accepted Oct. 22, 2013; Crosschecked Dec. 16, 2013

Abstract: Exponential stability and robust exponential stability relating to switched systems consisting of stable and unstable nonlinear subsystems are considered in this study. At each switching time instant, the impulsive increments which are nonlinear functions of the states are extended from switched linear systems to switched nonlinear systems. Using the average dwell time method and piecewise Lyapunov function approach, when the total active time of unstable subsystems compared to the total active time of stable subsystems is less than a certain proportion, the exponential stability of the switched system is guaranteed. The switching law is designed which includes the average dwell time of the switched system. Switched systems with uncertainties are also studied. Sufficient conditions of the exponential stability and robust exponential stability are provided for switched nonlinear systems. Finally, simulations show the effectiveness of the result.

Key words: Average dwell time, Impulse, Exponential stability, Robustness
doi:10.1631/jzus.C1300123 **Document code:** A **CLC number:** TP13

1 Introduction

Switched systems are typical hybrid dynamical systems. Stability and robust stability for switched systems have been paid considerable attention in recent years. Switched systems consist of several subsystems and a switching law. The switching law decides which subsystem is active at each time instant. Switched systems will be unstable or demonstrate worse performance if an unsuitable switching law is chosen. Switched systems have extensive application backgrounds, for example, intelligent transportation systems (Varaiya, 1993), electrical power systems (Qin and Song, 2001), robot control systems (Hiskens, 2001), and chemical process control systems (Lennartson *et al*., 1996).

There are many results on stability of switched systems, especially for switched linear systems. The switched systems composed of nonlinear subsystems have more general engineering background and research significance. Due to a lack of efficient studies and designing tools, the results on switched systems composed of nonlinear subsystems are few and immature. If there exists a common Lyapunov function for every subsystem, the switched system can be stable under arbitrary switching laws. However, many switched systems do not have a common Lyapunov function. The average dwell time method (Hespanha and Morse, 1999; Liberzon, 2003) is an effective approach for switched systems without common Lyapunov functions. A type of switched system whose structural matrices have a stably convex combination was studied in Wicks *et al*. (1998). Kim *et al.* (2006) extended the result to switched systems with time delay and obtained sufficient conditions of system stability. The above two results were generalized to switched systems with time-varying delay by linear matrix inequality (LMI) in Sun *et al*. (2008), and a less conservative condition was attained. Zhai *et al*. (2001a) considered the disturbance attenuation of

* Project supported by the National Natural Science Foundation of China (Nos. 61074004 and 61374037)

switched systems consisting of several linear time-invariant subsystems. Using the average dwell time method and piecewise Lyapunov function approach, a sufficient condition of switched systems satisfying the level of disturbance attenuation less than γ_0 was obtained, where each subsystem was Hurwitz stable and had the level of disturbance attenuation less than γ_0. When not all subsystems were Hurwitz stable, and the total active time of unstable subsystems compared to the total active time of stable subsystems was small enough, the sufficient condition of the switched system satisfying the level of disturbance attenuation less than γ_0 also held. The stability of switched linear systems which were composed of Hurwitz stable subsystems and unstable subsystems was discussed in Zhai *et al*. (2001b). The switched systems were exponentially stable with designated margins under the constructed switching law. The stability of the switched systems with nonlinear disturbances was also studied. Zhai *et al*. (2005) investigated the stability and L_2-gain performance of switched linear systems which were made up of continuous-time and discrete-time subsystems. All the continuous-time subsystems were Hurwitz stable, and all the discrete-time subsystems were Schur stable. All subsystems satisfied the L_2-gain performance less than a given positive number γ. A sufficient condition of the stability and L_2-gain performance less than the given positive number γ of the switched linear system under an arbitrary switching law was given, when every subsystem had a common Lyapunov function. When every subsystem did not have a common Lyapunov function, but a piecewise Lyapunov function, a sufficient condition of the stability and weighting L_2-gain performance which was less than the given positive number γ of the switched linear system was obtained using an average dwell time method. Sun *et al*. (2007) studied the stability and L_2-gain performance of switched linear systems composed of stable and unstable subsystems with time-varying delay and disturbances. When the total active time of unstable subsystems compared to the total active time of stable subsystems was small enough, the stability and L_2-gain performance of the switched system could be guaranteed with the given condition.

On the other hand, the occurring impulse in the system has an extensively practical background. If the impulse cannot be handled well, it will affect the stability and robust design of the switched system. Linear systems with multiple delays were studied in de la Sen and Luo (2003). The systems had inputs of impulses generated at different time instants. Compared to the delay dynamics, if the delay-free part was stable enough, the amplitudes of the impulses can be selected larger and the time intervals between them can be selected longer. When the delay-free dynamics were not exponentially stable, the systems could admit faster input impulses and signs to achieve stabilization. Marchenko and Zaczkiewicz (2009) considered a kind of hybrid dynamical system with the effect of impulses, including both continuous and discrete components. Using the Cauchy formula, the solutions of its dual systems were obtained. Then the integral representations of the solutions were derived by the solutions of its dual systems. The time-varying linear systems were considered in de la Sen (2006). The systems had a time-varying delay, and the delay-free dynamics and the delays were time-varying and impulsive. The input from outside could also be impulsive. Using three different auxiliary homogeneous systems, the state and output trajectories were constructed with given bounded initial conditions. The stability of the system was analyzed. Most of the results on impulsive increments are linear functions of the states (Xu *et al*., 2005; 2008; Yao *et al*., 2006). Xu and Teo (2010) investigated the exponential stability of switched systems with impulsive increments which were nonlinear functions of the states. A sufficient condition of the exponential stability for impulsive switched systems was given. However, every subsystem was a linear system. Zong *et al*. (2008) discussed the robust stabilization of uncertain impulsive switched systems with time delay in various states. The impulsive increments were linear functions of the states. They studied two cases: (1) the time delay was larger than the dwell time; (2) the time delay was less than the dwell time. A sufficient condition of globally asymptotical stability of an impulsive switched system was given. Hespanha *et al*. (2008) considered the input-to-state stability (ISS) of impulsive systems. When the continuous state was input-to-state stable and the discrete state was input-to-state unstable, or both dynamics are input-to-state stable, sufficient conditions of ISS of the system were given by using the average dwell time method. Liu and Marquez (2008) discussed the controllability and observability of switching impulsive systems. By using the characteristic polynomial theory of matrix, a necessary and sufficient condition for controllability and controlled observability was given. Zhu (2010) investigated the stability of switched impulsive systems with time delay. Not all subsystems were necessarily stable. An easily verifiable sufficient condition of switched impulsive systems was given using the minimum holding time. Liu *et al*. (2011) studied ISS and the integral input-to-state stability (iISS) of impulsive and switching hybrid systems with time delay using a Lyapunov-Krasovskii function. With the impulsive increments, when the impulse and switching signal satisfied the upper bound of the average dwell time, the whole switched system would be input-to-state stable (or integral input-to-state stable) even if none of the subsystems was input-to-state stable (or integral input-to-state stable). They also investigated the case when every subsystem was input-to-state stable (or integral input-to-state stable). The whole switched system would be input-to-state stable (or integral input-to-state stable) if the impulse and switching signal satisfied the lower bound of the average dwell time. Wang *et al*. (2008) studied the stabilization and L_2-gain performance of uncertain switched cascade systems. The uncertainties were norm bounded time-varying. By applying the average dwell time method and piecewise Lyapunov function method, switched nonlinear systems were stable and had an L_2-gain performance with suitable designed controllers as long as every subsystem was stabilized. They did not discuss the case when any subsystems of the switched system were unstable, or include the impulsive increments at the switching time instant.

To our knowledge, there is no result on the switched cascade nonlinear systems with nonlinear impulsive increments. In this paper, the exponential stability and robust exponential stability of switched systems consisting of stable and unstable nonlinear subsystems are considered. The impulsive increments which are nonlinear functions of the states are considered at each switching time instant. Using the average dwell time method and piecewise Lyapunov function approach, if the total active time of unstable subsystems compared to the total active time of stable subsystems is less than a certain proportion, the exponential stability of the switched system is guaranteed. Switched systems with uncertainties are also studied. Sufficient conditions of the exponential stability and robust exponential stability are provided. We extend the results of nonlinear impulsive increments for switched linear systems in Xu and Teo (2010) to those for switched nonlinear systems. Finally, simulations show the effectiveness of the extension.

2 System description

Consider the following switched nonlinear system consisting of m subsystems

$$\begin{cases} \dot{\boldsymbol{x}}_1(t) = \boldsymbol{A}_{1\sigma(t)}\boldsymbol{x}_1(t) + \boldsymbol{A}_{2\sigma(t)}\boldsymbol{x}_2(t), & \multirow{2}{*}{$t \neq t_k$,} \\ \dot{\boldsymbol{x}}_2(t) = \boldsymbol{f}_{2\sigma(t)}(\boldsymbol{x}_2(t)), & t \neq t_k, \\ \Delta\boldsymbol{x}_1(t) = \boldsymbol{E}_{1k}\boldsymbol{x}_1(t) + \boldsymbol{\phi}_{1k}(t, \boldsymbol{x}_1(t)), & t = t_k, \\ \Delta\boldsymbol{x}_2(t) = \boldsymbol{E}_{2k}\boldsymbol{x}_2(t) + \boldsymbol{\phi}_{2k}(t, \boldsymbol{x}_2(t)), & t = t_k, \end{cases} \tag{1}$$

where $\boldsymbol{x}(t)\in\mathbb{R}^n$ is the state of the system, $\boldsymbol{x}_1(t)\in\mathbb{R}^q$ and $\boldsymbol{x}_2(t)\in\mathbb{R}^{n-q}$ are the linear and nonlinear components with appropriate dimensions, respectively, $\sigma(t)$: $[0, +\infty)\rightarrow M=\{1, 2, \ldots, m\}$ is the switching signal, and m is a positive integer denoting the number of subsystems. Corresponding to $\sigma(t)$, we have the switching sequence

$$\{\boldsymbol{x}_{t_0}; (i_0, t_0), \ldots, (i_k, t_k), \ldots, \mid i_k\in M, k=0, 1, \ldots\},$$

which means that when $t\in(t_k, t_{k+1}]$, the i_kth subsystem is activated. $\boldsymbol{A}_{1i}$ and $\boldsymbol{A}_{2i}$ are constant matrices with appropriate dimensions; $\boldsymbol{\phi}_{1k}(t, \boldsymbol{x}_1(t))$ and $\boldsymbol{\phi}_{2k}(t, \boldsymbol{x}_2(t))$ are nonlinear vector-valued functions of t and the components $\boldsymbol{x}_1(t)$ or $\boldsymbol{x}_2(t)$ of the state, respectively, satisfying $\boldsymbol{\phi}_{1k}(t, \boldsymbol{0})\equiv\boldsymbol{\phi}_{2k}(t, \boldsymbol{0})\equiv\boldsymbol{0}$ for all of $t\in[t_0, +\infty)$; $\boldsymbol{f}_{2i}(\boldsymbol{x}_2(t))$ is the nonlinear smooth vector field for each certain i, and satisfies $\boldsymbol{f}_{2i}(0)=\boldsymbol{0}$; $\Delta\boldsymbol{x}_1(t)$ and $\Delta\boldsymbol{x}_2(t)$ are the impulsive increments at each switching time instant of the switched system corresponding to the state components $\boldsymbol{x}_1(t)$ and $\boldsymbol{x}_2(t)$.

$$\Delta\boldsymbol{x}_1(t)=\boldsymbol{x}_1(t_k^+)-\boldsymbol{x}_1(t_k^-)=\boldsymbol{x}_1(t_k^+)-\boldsymbol{x}_1(t_k),$$
$$\Delta\boldsymbol{x}_2(t)=\boldsymbol{x}_2(t_k^+)-\boldsymbol{x}_2(t_k^-)=\boldsymbol{x}_2(t_k^+)-\boldsymbol{x}_2(t_k),$$

where

$$\boldsymbol{x}_1(t_k^+)=\lim_{t\downarrow t_k}\boldsymbol{x}_1(t),\ \boldsymbol{x}_2(t_k^+)=\lim_{t\downarrow t_k}\boldsymbol{x}_2(t),$$
$$\boldsymbol{x}_1(t_k)=\boldsymbol{x}_1(t_k^-)=\lim_{t\uparrow t_k}\boldsymbol{x}_1(t),\ \boldsymbol{x}_2(t_k)=\boldsymbol{x}_2(t_k^-)=\lim_{t\uparrow t_k}\boldsymbol{x}_2(t).$$

It means that the solutions of the nonlinear impulsive switched system (1) are left-continuous. $\boldsymbol{E}_{1k}$ and $\boldsymbol{E}_{2k}$ are constant matrices with appropriate dimensions. We assume that the impulsive effects are coupled with the switching signal.

If the switched system contains uncertainties, the system can be expressed as

$$\begin{cases} \dot{\boldsymbol{x}}_1(t) = (\boldsymbol{A}_{1\sigma(t)} + \Delta\boldsymbol{A}_{1\sigma(t)})\boldsymbol{x}_1(t) \\ \qquad + (\boldsymbol{A}_{2\sigma(t)} + \Delta\boldsymbol{A}_{2\sigma(t)})\boldsymbol{x}_2(t), \quad t \neq t_k, \\ \dot{\boldsymbol{x}}_2(t) = (\boldsymbol{f}_{2\sigma(t)} + \Delta\boldsymbol{f}_{2\sigma(t)})(\boldsymbol{x}_2(t)), \\ \Delta\boldsymbol{x}_1(t) = \boldsymbol{E}_{1k}\boldsymbol{x}_1(t) + \boldsymbol{\phi}_{1k}(t, \boldsymbol{x}_1(t)), \\ \Delta\boldsymbol{x}_2(t) = \boldsymbol{E}_{2k}\boldsymbol{x}_2(t) + \boldsymbol{\phi}_{2k}(t, \boldsymbol{x}_2(t)), \quad t = t_k, \end{cases} \tag{2}$$

where $\Delta\boldsymbol{A}_{1i}$, $\Delta\boldsymbol{A}_{2i}$, and $\Delta\boldsymbol{f}_{2i}$ are the uncertainties of the ith subsystem.

In this study, we use the standard notations: '$\|\cdot\|$' denotes the Euclidean norm of a vector or a matrix; $\boldsymbol{P}>\boldsymbol{0}$ ($\geq\boldsymbol{0}$, $<\boldsymbol{0}$, or $\leq\boldsymbol{0}$) represents that matrix $\boldsymbol{P}$ is positive definite (positive semidefinite, negative definite, or negative semidefinite); $\lambda_{\max}(\boldsymbol{R})$ and $\lambda_{\min}(\boldsymbol{R})$ denote the maximum and minimum eigenvalues of a symmetric matrix $\boldsymbol{R}$, respectively.

To obtain the exponential stability and robust exponential stability of nonlinear impulsive switched systems with stable and unstable subsystems, the following assumptions, definitions, and lemmas are required.

Assumption 1 The time-varying structural uncertainties have the following form:

$$[\Delta\boldsymbol{A}_{1i}(t), \Delta\boldsymbol{A}_{2i}(t)]=\boldsymbol{D}_i\boldsymbol{F}_i(t)[\boldsymbol{H}_{1i}, \boldsymbol{H}_{2i}], \tag{3}$$

where $\boldsymbol{D}_i$, $\boldsymbol{H}_{1i}$, and $\boldsymbol{H}_{2i}$ are constant matrices with appropriate dimensions, and $\boldsymbol{F}_i(t)$ is an unknown time-varying matrix satisfying $\boldsymbol{F}_i^{\mathrm{T}}(t)\boldsymbol{F}_i(t)\leq\boldsymbol{I}$ $\forall t$. $\boldsymbol{I}$ is a unit matrix with an appropriate dimensionality.

Assumption 2 The uncertainty in nonlinear component $\boldsymbol{x}_2(t)$ satisfies

$$\|\Delta\boldsymbol{f}_{2i}(\boldsymbol{x}_2)\| \leq q_i \|\boldsymbol{f}_{2i}(\boldsymbol{x}_2)\| \tag{4}$$

for every $i\in M$, where $0\leq q_i\leq 1$.

Assumption 3 The nonlinear impulsive increments satisfy

$$\begin{cases} \|\boldsymbol{\phi}_{1k}(t, \boldsymbol{x}_1(t))\| \leq \rho_{1k} \|\boldsymbol{x}_1(t)\|, \\ \|\boldsymbol{\phi}_{2k}(t, \boldsymbol{x}_2(t))\| \leq \rho_{2k} \|\boldsymbol{x}_2(t)\|, \end{cases} \tag{5}$$

for all $t\in[t_0, +\infty)$, where $\rho_{1k}=\rho(\boldsymbol{I}+\boldsymbol{E}_{1k})$ and $\rho_{2k}=\rho(\boldsymbol{I}+\boldsymbol{E}_{2k})$. $\rho(\cdot)$ denotes the spectral radius of the matrix in the bracket.

Definition 1 The equilibrium $\boldsymbol{x}^*=\boldsymbol{0}$ of system (1) is said to be exponentially stable under the given switching law $\sigma(t)$, if the solution of system (1) satisfies

$$\|\boldsymbol{x}(t)\|\leq\kappa\boldsymbol{x}(t_0)\mathrm{e}^{-\lambda(t-t0)},\ \forall t\geq t_0,$$

for the given positive real numbers λ and κ, where $\boldsymbol{x}(t_0)$ is the initial state of system (1) at $t=t_0$.

Definition 2 (Hespanha and Morse, 1999) For any $T_2>T_1\geq 0$, let $N_\sigma(T_1, T_2)$ denote the number of switching signal $\sigma(t)$ over time interval (T_1, T_2). If $N_\sigma(T_1, T_2)\leq N_0+(T_2-T_1)/T_{\mathrm{a}}$ holds for $T_{\mathrm{a}}>0$, $N_0\geq 0$, then T_{a} is called the average dwell time. As commonly used in the literature, we choose the chatter bound $N_0=0$.

Lemma 1 (Petersen and Hollot, 1986) For given matrices $\boldsymbol{Q}=\boldsymbol{Q}^{\mathrm{T}}<\boldsymbol{0}$, $\boldsymbol{H}$, $\boldsymbol{E}$, and $\boldsymbol{R}=\boldsymbol{R}^{\mathrm{T}}>\boldsymbol{0}$ with appropriate dimensions,

$$\boldsymbol{Q}+\boldsymbol{H}\boldsymbol{F}(t)\boldsymbol{E}+\boldsymbol{E}^{\mathrm{T}}\boldsymbol{F}^{\mathrm{T}}(t)\boldsymbol{H}^{\mathrm{T}}<\boldsymbol{0} \tag{6a}$$

holds for any $\boldsymbol{F}(t)$ satisfying $\boldsymbol{F}^{\mathrm{T}}(t)\boldsymbol{F}(t)\leq\boldsymbol{R}$ if and only if there exists a positive real number ε such that

$$\boldsymbol{Q}+\varepsilon\boldsymbol{H}\boldsymbol{H}^{\mathrm{T}}+\varepsilon^{-1}\boldsymbol{E}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{E}<\boldsymbol{0}. \tag{6b}$$

Lemma 2 (Guan *et al.*, 2005) Let $\boldsymbol{P}\in\mathbb{R}^{n\times n}$ be a given symmetric positive definite matrix and let $\boldsymbol{Q}\in\mathbb{R}^{n\times n}$ be a given symmetric matrix. Then

$$\lambda_{\min}(\boldsymbol{P}^{-1}\boldsymbol{Q})\Omega(t)\leq\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{Q}\boldsymbol{x}(t)\leq\lambda_{\max}(\boldsymbol{P}^{-1}\boldsymbol{Q})\Omega(t) \tag{7}$$

for all $\boldsymbol{x}(t)\in\mathbb{R}^n$, where $\Omega(t)=\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{x}(t)$.

Remark 1 The subsystem of the switched system (1) is a kind of cascade nonlinear system. When a nonlinear system cannot be exactly linearized, it has the form of subsystem (1). So, this form has some certain representatives.

3 Main results

3.1 Exponential stability for nonlinear impulsive switched systems with stable and unstable subsystems

For the subsystems of the switched system (1), we assume that every subsystem is stable when $i\in M_{\mathrm{s}}=\{h_1, h_2, \ldots, h_r\}$, and every subsystem is unstable when $i\in M_u=\{h_{r+1}, h_{r+2}, \ldots, h_m\}$, where $h_i\in M$ ($1\le i\le m$). Thus, the following results can be obtained:

Lemma 3 For every $i\in M_s$, the stable nonlinear subsystem

$$\begin{cases}\dot{\boldsymbol{x}}_1(t)=\boldsymbol{A}_{1i}\boldsymbol{x}_1(t)+\boldsymbol{A}_{2i}\boldsymbol{x}_2(t),\\ \dot{\boldsymbol{x}}_2(t)=\boldsymbol{f}_{2i}(\boldsymbol{x}_2(t)),\end{cases}\tag{8}$$

satisfies that if there are positive numbers λ^- and ε_i such that

(i) There exists a positive definite solution matrix $\boldsymbol{P}_i$ leading to

$$\begin{pmatrix}\boldsymbol{A}_{1i}^{\mathrm{T}}\boldsymbol{P}_i+\boldsymbol{P}_i\boldsymbol{A}_{1i}+\lambda^-\boldsymbol{P}_i & \boldsymbol{P}_i\\ \boldsymbol{P}_i & -\varepsilon_i\boldsymbol{I}\end{pmatrix}<\boldsymbol{0}.\tag{9a}$$

(ii) There exists a proper, positive definite, radically unbounded function $W_i(\boldsymbol{x}_2)$, such that

$$\frac{\partial W_i(\boldsymbol{x}_2)}{\partial\boldsymbol{x}_2}\cdot\boldsymbol{f}_{2i}(\boldsymbol{x}_2)\le -b_{1i}\|\boldsymbol{x}_2\|^2,\tag{9b}$$

$$a_{1i}\|\boldsymbol{x}_2\|^2\le W_i(\boldsymbol{x}_2)\le a_{2i}\|\boldsymbol{x}_2\|^2,\tag{9c}$$

where a_{1i}, a_{2i}, and b_{1i} are all positive numbers. Then the state of system (8) from time t_0 satisfies

$$V_i(t)\le \mathrm{e}^{-\lambda^-(t-t_0)}V_i(t_0),$$

where $V_i(t)$ is a Lyapunov function candidate of system (8) with

$$V_i(t)=\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{P}_i\boldsymbol{x}_1+l_iW_i(\boldsymbol{x}_2).\tag{10}$$

l_i is a constant to be specified in the proof.

Proof For the stable subsystem (8), we choose $V_i(t)$ in Eq. (10) as the Lyapunov function candidate. Then the time derivative of $V_i(t)$ along the trajectory of subsystem (8) is

$$\begin{aligned}\dot V_i(\boldsymbol{x}(t))&=\dot{\boldsymbol{x}}_1^{\mathrm{T}}\boldsymbol{P}_i\boldsymbol{x}_1+\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{P}_i\dot{\boldsymbol{x}}_1+l_i\frac{\partial W_i}{\partial\boldsymbol{x}_2}(\boldsymbol{x}_2)\cdot\boldsymbol{f}_{2i}(\boldsymbol{x}_2)\\ &=(\boldsymbol{A}_{1i}\boldsymbol{x}_1+\boldsymbol{A}_{2i}\boldsymbol{x}_2)^{\mathrm{T}}\boldsymbol{P}_i\boldsymbol{x}_1+\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{P}_i(\boldsymbol{A}_{1i}\boldsymbol{x}_1+\boldsymbol{A}_{2i}\boldsymbol{x}_2)\\ &\quad+l_i\frac{\partial W_i}{\partial\boldsymbol{x}_2}(\boldsymbol{x}_2)\cdot\boldsymbol{f}_{2i}(\boldsymbol{x}_2)\\ &\le\boldsymbol{x}_1^{\mathrm{T}}(\boldsymbol{A}_{1i}^{\mathrm{T}}\boldsymbol{P}_i+\boldsymbol{P}_i\boldsymbol{A}_{1i})\boldsymbol{x}_1+2\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{P}_i\boldsymbol{A}_{2i}\boldsymbol{x}_2-l_ib_{1i}\|\boldsymbol{x}_2\|^2.\end{aligned}\tag{11}$$

From Lemma 1, there is a positive number ε_i such that

$$2\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{P}_i\boldsymbol{A}_{2i}\boldsymbol{x}_2\le\varepsilon_i\boldsymbol{x}_2^{\mathrm{T}}\boldsymbol{A}_{2i}^{\mathrm{T}}\boldsymbol{A}_{2i}\boldsymbol{x}_2+\varepsilon_i^{-1}\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{P}_i\boldsymbol{P}_i\boldsymbol{x}_1.\tag{12a}$$

Using the Schur complementary lemma, that condition (i) in Lemma 3 holds is equivalent to

$$\boldsymbol{A}_{1i}^{\mathrm{T}}\boldsymbol{P}_i+\boldsymbol{P}_i\boldsymbol{A}_{1i}+\lambda^-\boldsymbol{P}_i+\varepsilon_i^{-1}\boldsymbol{P}_i\boldsymbol{P}_i<\boldsymbol{0}.\tag{12b}$$

According to Eqs. (12a) and (12b), Eq. (11) can be changed to

$$\begin{aligned}&\dot V_i(\boldsymbol{x}(t))\le-\lambda^-\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{P}_i\boldsymbol{x}_1+\varepsilon_i\boldsymbol{x}_2^{\mathrm{T}}\boldsymbol{A}_{2i}^{\mathrm{T}}\boldsymbol{A}_{2i}\boldsymbol{x}_2-l_ib_{1i}\|\boldsymbol{x}_2\|^2\\ &\le-\lambda^-[\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{P}_i\boldsymbol{x}_1+l_iW_i(\boldsymbol{x}_2)]+\lambda^-l_iW_i(\boldsymbol{x}_2)\\ &\quad+\varepsilon_i\lambda_{\max}(\boldsymbol{A}_{2i}^{\mathrm{T}}\boldsymbol{A}_{2i})\boldsymbol{x}_2^{\mathrm{T}}\boldsymbol{x}_2-l_ib_{1i}\|\boldsymbol{x}_2\|^2\\ &\le-\lambda^-V_i(\boldsymbol{x})+\lambda^-l_ia_{2i}\|\boldsymbol{x}_2\|^2+\varepsilon_i\lambda_{\max}(\boldsymbol{A}_{2i}^{\mathrm{T}}\boldsymbol{A}_{2i})\boldsymbol{x}_2^{\mathrm{T}}\boldsymbol{x}_2\\ &\quad-l_ib_{1i}\|\boldsymbol{x}_2\|^2\\ &\le-\lambda^-V_i(\boldsymbol{x})-[l_i(b_{1i}-\lambda^-a_{2i})+\varepsilon_i\lambda_{\max}(\boldsymbol{A}_{2i}^{\mathrm{T}}\boldsymbol{A}_{2i})]\|\boldsymbol{x}_2\|^2.\end{aligned}\tag{12c}$$

We can choose

$$l_i\le\varepsilon_i\lambda_{\max}(\boldsymbol{A}_{2i}^{\mathrm{T}}\boldsymbol{A}_{2i})/(\lambda^-a_{2i}-b_{1i}),\ (i\in M_s),$$

and thus

$$\begin{aligned}&\dot V_i(\boldsymbol{x}(t))\le-\lambda^-V_i(\boldsymbol{x}(t))\\ &\Rightarrow V_i(\boldsymbol{x}(t))\le V_i(\boldsymbol{x}(t_0))\mathrm{e}^{-\lambda^-(t-t_0)}.\end{aligned}\tag{13}$$

Lemma 4 For every $i\in M_u$, the unstable nonlinear subsystem

$$\begin{cases}\dot{\boldsymbol{x}}_1(t)=\boldsymbol{A}_{1i}\boldsymbol{x}(t)+\boldsymbol{A}_{2i}\boldsymbol{x}_2(t),\\ \dot{\boldsymbol{x}}_2(t)=\boldsymbol{f}_{2i}(\boldsymbol{x}_2(t)),\end{cases}\tag{14}$$

satisfies that if there exists positive numbers λ^+ and ε_i such that

(ia) There exists a positive definite solution matrix $\boldsymbol{P}_i$ leading to

$$\begin{pmatrix}\boldsymbol{A}_{1i}^{\mathrm{T}}\boldsymbol{P}_i+\boldsymbol{P}_i\boldsymbol{A}_{1i}-\lambda^+\boldsymbol{P}_i & \boldsymbol{P}_i\\ \boldsymbol{P}_i & -\varepsilon_i\boldsymbol{I}\end{pmatrix}<\boldsymbol{0}.\tag{15a}$$

(iia) There exists a proper, positive definite, and radically unbounded function $W_i(\boldsymbol{x}_2)$, such that

$$\frac{\partial W_i(\boldsymbol{x}_2)}{\partial\boldsymbol{x}_2}\cdot\boldsymbol{f}_{2i}(\boldsymbol{x}_2)\le b_{2i}\|\boldsymbol{x}_2\|^2,\tag{15b}$$

$$a_{1i}\|\boldsymbol{x}_2\|^2 \le W_i(\boldsymbol{x}_2) \le a_{2i}\|\boldsymbol{x}_2\|^2, \tag{15c}$$

where a_{1i}, a_{2i}, and b_{2i} are all positive numbers. Then the states of system (14) from time t_0 satisfy

$$V_i(t) \le \mathrm{e}^{\lambda^+(t-t_0)}V_i(t_0),$$

where $V_i(t)=\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{P}_i\boldsymbol{x}_1+l_iW_i(\boldsymbol{x}_2)$ is a Lyapunov function candidate of system (14) with l_i being a constant to be specified in the proof.

Proof For the unstable subsystem (14), we choose the following Lyapunov function candidate:

$$V_i(t)=\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{P}_i\boldsymbol{x}_1+l_iW_i(\boldsymbol{x}_2).$$

Then the time derivative of $V_i(t)$ along the trajectory of subsystem (14) is

$$\begin{aligned}\dot{V}_i(\boldsymbol{x}(t)) &= \dot{\boldsymbol{x}}_1^{\mathrm{T}}\boldsymbol{P}_i\boldsymbol{x}_1+\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{P}_i\dot{\boldsymbol{x}}_1+l_i\frac{\partial W_i}{\partial \boldsymbol{x}_2}(\boldsymbol{x}_2)\cdot \boldsymbol{f}_{2i}(\boldsymbol{x}_2)\\ &=(\boldsymbol{A}_{1i}\boldsymbol{x}_1+\boldsymbol{A}_{2i}\boldsymbol{x}_2)^{\mathrm{T}}\boldsymbol{P}_i\boldsymbol{x}_1+\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{P}_i(\boldsymbol{A}_{1i}\boldsymbol{x}_1+\boldsymbol{A}_{2i}\boldsymbol{x}_2)\\ &\quad+l_i\frac{\partial W_i}{\partial \boldsymbol{x}_2}(\boldsymbol{x}_2)\cdot \boldsymbol{f}_{2i}(\boldsymbol{x}_2)\\ &=\boldsymbol{x}_1^{\mathrm{T}}(\boldsymbol{A}_{1i}^{\mathrm{T}}\boldsymbol{P}_i+\boldsymbol{P}_i\boldsymbol{A}_{1i})\boldsymbol{x}_1+2\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{P}_i\boldsymbol{A}_{2i}\boldsymbol{x}_2+l_ib_{2i}\|\boldsymbol{x}_2\|^2.\end{aligned} \tag{16}$$

Similar to Eq. (12a), there is a positive real number ε_i such that

$$2\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{P}_i\boldsymbol{A}_{2i}\boldsymbol{x}_2 \le \varepsilon_i\boldsymbol{x}_2^{\mathrm{T}}\boldsymbol{A}_{2i}^{\mathrm{T}}\boldsymbol{A}_{2i}\boldsymbol{x}_2+\varepsilon_i^{-1}\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{P}_i\boldsymbol{P}_i\boldsymbol{x}_1. \tag{17a}$$

From Eq. (16), the following inequality can be obtained:

$$\begin{aligned}\dot{V}_i(\boldsymbol{x}(t)) &\le \boldsymbol{x}_1^{\mathrm{T}}(\boldsymbol{A}_{1i}^{\mathrm{T}}\boldsymbol{P}_i+\boldsymbol{P}_i\boldsymbol{A}_{1i})\boldsymbol{x}_1+\varepsilon_i\boldsymbol{x}_2^{\mathrm{T}}\boldsymbol{A}_{2i}^{\mathrm{T}}\boldsymbol{A}_{2i}\boldsymbol{x}_2\\ &\quad+\varepsilon_i^{-1}\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{P}_i\boldsymbol{P}_i\boldsymbol{x}_1+l_ib_{2i}\|\boldsymbol{x}_2\|^2.\end{aligned}$$

Using the Schur complementary lemma, that condition (ia) in Lemma 4 holds is equivalent to

$$\boldsymbol{A}_{1i}^{\mathrm{T}}\boldsymbol{P}_i+\boldsymbol{P}_i\boldsymbol{A}_{1i}-\lambda^+\boldsymbol{P}_i+\varepsilon_i^{-1}\boldsymbol{P}_i\boldsymbol{P}_i<\boldsymbol{0}. \tag{17b}$$

Therefore, according to Eqs. (17a) and (17b), Eq. (16) can be changed to

$$\begin{aligned}\dot{V}_i(\boldsymbol{x}(t)) &\le \lambda^+\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{P}_i\boldsymbol{x}_1+\varepsilon_i\boldsymbol{x}_2^{\mathrm{T}}\boldsymbol{A}_{2i}^{\mathrm{T}}\boldsymbol{A}_{2i}\boldsymbol{x}_2+l_ib_{2i}\|\boldsymbol{x}_2\|^2\\ &\le \lambda^+(\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{P}_i\boldsymbol{x}_1+l_iW_i(\boldsymbol{x}_2))-\lambda^+l_iW_i(\boldsymbol{x}_2)\\ &\quad+\varepsilon_i\lambda_{\max}(\boldsymbol{A}_{2i}^{\mathrm{T}}\boldsymbol{A}_{2i})\boldsymbol{x}_2^{\mathrm{T}}\boldsymbol{x}_2+l_ib_{2i}\|\boldsymbol{x}_2\|^2\\ &\le \lambda^+V_i(\boldsymbol{x})-\lambda^+l_ia_{1i}\|\boldsymbol{x}_2\|^2+\varepsilon_i\lambda_{\max}(\boldsymbol{A}_{2i}^{\mathrm{T}}\boldsymbol{A}_{2i})\|\boldsymbol{x}_2\|^2\\ &\quad+l_ib_{2i}\|\boldsymbol{x}_2\|^2\\ &=\lambda^+V_i(\boldsymbol{x})-[l_i(\lambda^+a_{1i}-b_{2i})-\varepsilon_i\lambda_{\max}(\boldsymbol{A}_{2i}^{\mathrm{T}}\boldsymbol{A}_{2i})]\|\boldsymbol{x}_2\|^2.\end{aligned} \tag{17c}$$

We can choose

$$l_i \ge \varepsilon_i\lambda_{\max}(\boldsymbol{A}_{2i}^{\mathrm{T}}\boldsymbol{A}_{2i})/(\lambda^+a_{1i}-b_{2i}),\ (i\in M_{\mathrm{u}}),$$

and thus

$$\dot{V}_i(x(t)) \le \lambda^+V_i(x(t)) \Rightarrow V_i(x(t)) \le V_i(x(t_0))\mathrm{e}^{\lambda^+(t-t_0)}.$$

Similar to Sun *et al.* (2007), we select the following notations: For any switching signal and any $0\le t_1<t_2$, let $T^+(t_1, t_2)$ and $T^-(t_1, t_2)$ denote the total active time of the unstable and the stable subsystems during time interval (t_1, t_2), respectively.

We give the switching law as

S1: For switching signal $\sigma(t)$, if there exists subsequence $p_0=t_0<p_1<p_2<\ldots$ of time series $t_0<t_1<t_2<\ldots$, and for every time interval $[p_k, p_{k+1})$, there is

$$\frac{T^-(p_k, p_{k+1})}{T^+(p_k, p_{k+1})} \ge \frac{\lambda^+ +\omega}{\lambda^- -\omega}, \tag{18}$$

where $\omega\in[0, \lambda^-)$. Let $\max\{p_{k+1}-p_k\}\le\delta<+\infty$, $\forall k\in\{0, 1, \ldots\}$. $\bar{\delta}$ and δ^* are given positive numbers satisfying $\ln\beta-\bar{\delta}<0$. We assume that the average dwell time T_{a} satisfies the following inequality

$$T_{\mathrm{a}} \ge \frac{\bar{\delta}}{\omega^*},\quad \omega^*\in(\delta^*, \omega), \tag{19}$$

where β is the upper bound of the impulsive increments during the switching time for the switched system. Its concrete form is given in Theorem 1.

A sufficient condition of the exponential stability for the nonlinear impulsive switched system (1) with stable and unstable subsystems is also given by Theorem 1.

Theorem 1 For the nonlinear impulsive switched system (1) with stable and unstable subsystems, when $i\in M_{\mathrm{s}}$, the subsystems are stable and meet conditions (i) and (ii) in Lemma 3; When $i\in M_{\mathrm{u}}$, the subsystems are unstable and meet conditions (ia) and (iia) in

Lemma 4. The nonlinear impulsive increments satisfy Assumption 3, the switching law is carried out according to S1, and then the switched system is exponentially stable. The attenuation of the states satisfies

$$\|\boldsymbol{x}(t)\| \le \sqrt{k_2 \mathrm{e}^{c} / k_1}\, \|\boldsymbol{x}(t_0)\| \mathrm{e}^{-\eta(t-t_0)}, \tag{20}$$

where

$$\begin{aligned}
&\beta = \max\{\beta_1, \beta_2, ..., \beta_k\},\ \beta_k = \max\{\beta_{1k}, \beta_{2k}\},\\
&\beta_{1k} = \tilde{\lambda}_{1k} + \rho_{1k}^2 \lambda_{\max}(\boldsymbol{P}_{i_k}\boldsymbol{P}_{i_{k-1}}^{-1}) + 2\sqrt{\tilde{\lambda}_{1k}\rho_{1k}^2 \lambda_{\max}(\boldsymbol{P}_{i_k}\boldsymbol{P}_{i_{k-1}}^{-1})},\\
&\tilde{\lambda}_{1k} = \lambda_{\max}(\boldsymbol{P}_{i_{k-1}}^{-1}(\boldsymbol{I}+\boldsymbol{E}_{1k})^{\mathrm{T}}\boldsymbol{P}_{i_k}(\boldsymbol{I}+\boldsymbol{E}_{1k})),\\
&\beta_{2k} = \frac{l_{i_k} a_{2i_k}}{a_{1i_k}}\left(\tilde{\lambda}_{2k} + \rho_{2k}^2 + 2\sqrt{\tilde{\lambda}_{2k}\rho_{2k}^2}\right),\\
&\tilde{\lambda}_{2k} = \lambda_{\max}((\boldsymbol{I}+\boldsymbol{E}_{2k})^{\mathrm{T}}(\boldsymbol{I}+\boldsymbol{E}_{2k})),\ c = (\lambda^+ + \omega)\delta,\\
&k_1 = \min_i\{\lambda_{\min}(\boldsymbol{P}_i), l_i a_{1i}\},\ k_2 = \max_i\{\lambda_{\max}(\boldsymbol{P}_i), l_i a_{2i}\},\\
&\eta = (\omega - \omega^*)/2.
\end{aligned} \tag{21}$$

Proof We choose the piecewise Lyapunov function candidate as

$$V_{\sigma(t)}(t) = \boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{P}_{\sigma(t)}\boldsymbol{x}_1 + l_{\sigma(t)}W_{\sigma(t)}(\boldsymbol{x}_2). \tag{22}$$

When $t\in(t_k, t_{k+1}]$, the switched system switches to the i_kth subsystem; i.e., the i_kth subsystem is active during this time interval. Without loss of generality, we assume that the i_kth subsystem is stable. Then from Lemma 3, we obtain

$$V_{i_k}(\boldsymbol{x}(t)) \le V_{i_k}(\boldsymbol{x}(t_k))\mathrm{e}^{-\lambda^-(t-t_k)}.$$

When $t\in(t_{k+1}, t_{k+2}]$, the switched system switches to the i_{k+1}th subsystem. We assume that the i_{k+1}th subsystem is unstable. Then from Lemma 4, we obtain

$$V_{i_{k+1}}(\boldsymbol{x}(t)) \le V_{i_{k+1}}(\boldsymbol{x}(t_{k+1}))\mathrm{e}^{\lambda^+(t-t_{k+1})}.$$

At switching time instant t_k, the switched system switches from the i_{k-1}th subsystem to the i_kth subsystem with impulsive increment. Hence,

$$\begin{aligned}
V(\boldsymbol{x}(t_k^+)) &= [\boldsymbol{x}_1(t_k) + \Delta\boldsymbol{x}_1(t_k)]^{\mathrm{T}}\boldsymbol{P}_{i_k}[\boldsymbol{x}_1(t_k) + \Delta\boldsymbol{x}_1(t_k)]\\
&\quad + l_{i_k}W_{i_k}(\boldsymbol{x}_2(t_k) + \Delta\boldsymbol{x}_2(t_k))\\
&= V_1(\boldsymbol{x}(t_k^+)) + V_2(\boldsymbol{x}(t_k^+)),
\end{aligned} \tag{23}$$

where

$$V_1(\boldsymbol{x}(t_k^+)) = [\boldsymbol{x}_1(t_k) + \Delta\boldsymbol{x}_1(t_k)]^{\mathrm{T}}\boldsymbol{P}_{i_k}[\boldsymbol{x}_1(t_k) + \Delta\boldsymbol{x}_1(t_k)], \tag{24a}$$

$$V_2(\boldsymbol{x}(t_k^+)) = l_{i_k}W_{i_k}(\boldsymbol{x}_2(t_k) + \Delta\boldsymbol{x}_2(t_k)). \tag{24b}$$

By applying the Cauchy-Schwarz inequality for $V_1(\boldsymbol{x}(t_k^+))$, we obtain

$$\begin{aligned}
V_1(\boldsymbol{x}(t_k^+)) &= [(\boldsymbol{I}+\boldsymbol{E}_{1k})\boldsymbol{x}_1(t_k) + \boldsymbol{\phi}_{1k}(t_k, \boldsymbol{x}_1(t_k))]^{\mathrm{T}}\boldsymbol{P}_{i_k}\\
&\quad \cdot[(\boldsymbol{I}+\boldsymbol{E}_{1k})\boldsymbol{x}_1(t_k) + \boldsymbol{\phi}_{1k}(t_k, \boldsymbol{x}_1(t_k))]\\
&= \boldsymbol{x}_1^{\mathrm{T}}(t_k)(\boldsymbol{I}+\boldsymbol{E}_{1k})^{\mathrm{T}}\boldsymbol{P}_{i_k}(\boldsymbol{I}+\boldsymbol{E}_{1k})\boldsymbol{x}_1(t_k)\\
&\quad + 2\boldsymbol{x}_1^{\mathrm{T}}(t_k)(\boldsymbol{I}+\boldsymbol{E}_{1k})^{\mathrm{T}}\boldsymbol{P}_{i_k}\boldsymbol{\phi}_{1k}(t_k, \boldsymbol{x}_1(t_k))\\
&\quad + \boldsymbol{\phi}_{1k}^{\mathrm{T}}(t_k, \boldsymbol{x}_1(t_k))\boldsymbol{P}_{i_k}\boldsymbol{\phi}_{1k}(t_k, \boldsymbol{x}_1(t_k))\\
&\le \boldsymbol{x}_1^{\mathrm{T}}(t_k)(\boldsymbol{I}+\boldsymbol{E}_{1k})^{\mathrm{T}}\boldsymbol{P}_{i_k}(\boldsymbol{I}+\boldsymbol{E}_{1k})\boldsymbol{x}_1(t_k)\\
&\quad + 2\sqrt{\boldsymbol{x}_1^{\mathrm{T}}(t_k)(\boldsymbol{I}+\boldsymbol{E}_{1k})^{\mathrm{T}}\boldsymbol{P}_{i_k}(\boldsymbol{I}+\boldsymbol{E}_{1k})\boldsymbol{x}_1(t_k)}\\
&\quad \cdot\sqrt{\boldsymbol{\phi}_{1k}^{\mathrm{T}}(t_k, \boldsymbol{x}_1(t_k))\boldsymbol{P}_{i_k}\boldsymbol{\phi}_{1k}(t_k, \boldsymbol{x}_1(t_k))}\\
&\quad + \boldsymbol{\phi}_{1k}^{\mathrm{T}}(t_k, \boldsymbol{x}_1(t_k))\boldsymbol{P}_{i_k}\boldsymbol{\phi}_{1k}(t_k, \boldsymbol{x}_1(t_k))\\
&\le \tilde{\lambda}_{1k}V_1(t_k) + \rho_{1k}^2\lambda_{\max}\left(\boldsymbol{P}_{i_k}\boldsymbol{P}_{i_{k-1}}^{-1}\right)V_1(t_k)\\
&\quad + 2\sqrt{\tilde{\lambda}_{1k}V_1(t_k)\rho_{1k}^2\lambda_{\max}\left(\boldsymbol{P}_{i_k}\boldsymbol{P}_{i_{k-1}}^{-1}\right)V_1(t_k)}\\
&= \beta_{1k}V_1(t_k),
\end{aligned} \tag{25a}$$

$$\begin{aligned}
V_2(\boldsymbol{x}(t_k^+)) &= l_{i_k}W_{i_k}(\boldsymbol{x}_2(t_k) + \Delta\boldsymbol{x}_2(t_k))\\
&\le l_{i_k}a_{2i_k}\|\boldsymbol{x}_2(t_k) + \Delta\boldsymbol{x}_2(t_k)\|^2\\
&\le l_{i_k}a_{2i_k}[(\boldsymbol{I}+\boldsymbol{E}_{2k})\boldsymbol{x}_2(t_k) + \boldsymbol{\phi}_{2k}(t_k, \boldsymbol{x}_2(t_k))]^{\mathrm{T}}\\
&\quad \cdot[(\boldsymbol{I}+\boldsymbol{E}_{2k})\boldsymbol{x}_2(t_k) + \boldsymbol{\phi}_{2k}(t_k, \boldsymbol{x}_2(t_k))]\\
&= l_{i_k}a_{2i_k}\boldsymbol{x}_2^{\mathrm{T}}(t_k)(\boldsymbol{I}+\boldsymbol{E}_{2k})^{\mathrm{T}}(\boldsymbol{I}+\boldsymbol{E}_{2k})\boldsymbol{x}_2(t_k)\\
&\quad + 2l_{i_k}a_{2i_k}\boldsymbol{x}_2^{\mathrm{T}}(t_k)(\boldsymbol{I}+\boldsymbol{E}_{2k})^{\mathrm{T}}\boldsymbol{\phi}_{2k}(t_k, \boldsymbol{x}_2(t_k))\\
&\quad + l_{i_k}a_{2i_k}\boldsymbol{\phi}_{2k}^{\mathrm{T}}(t_k, \boldsymbol{x}_2(t_k))\boldsymbol{\phi}_{2k}(t_k, \boldsymbol{x}_2(t_k)).
\end{aligned} \tag{25b}$$

Using the Cauchy-Schwarz inequality for the above formula, we obtain

$$\begin{aligned}
V_2(\boldsymbol{x}(t_k^+)) &\le l_{i_k}a_{2i_k}\lambda_{\max}[(\boldsymbol{I}+\boldsymbol{E}_{2k})^{\mathrm{T}}(\boldsymbol{I}+\boldsymbol{E}_{2k})]\|\boldsymbol{x}_2(t_k)\|^2\\
&\quad + l_{i_k}a_{2i_k}\rho_{2i}^2\|\boldsymbol{x}_2(t_k)\|^2 + 2l_{i_k}a_{2i_k}\sqrt{\boldsymbol{x}_2^{\mathrm{T}}(t_k)(\boldsymbol{I}+\boldsymbol{E}_{2k})^{\mathrm{T}}}\\
&\quad \cdot\sqrt{(\boldsymbol{I}+\boldsymbol{E}_{2k})\boldsymbol{x}_2(t_k)\boldsymbol{\phi}_{2k}^{\mathrm{T}}(t_k, \boldsymbol{x}_2(t_k))\boldsymbol{\phi}_{2k}(t_k, \boldsymbol{x}_2(t_k))}\\
&\le \tilde{\lambda}_{2k}\frac{l_{i_k}a_{2i_k}}{a_{1i_k}}V_2(t_k) + \frac{l_{i_k}a_{2i_k}\rho_{2k}^2}{a_{1i_k}}V_2(t_k)\\
&\quad + 2\frac{l_{i_k}a_{2i_k}}{a_{1i_k}}\sqrt{\tilde{\lambda}_{2k}V_2(t_k)\rho_{2k}^2\,V_2(t_k)}\\
&= \beta_{2k}V_2(t_k).
\end{aligned} \tag{25c}$$

Therefore, we can obtain

$$V(x(t_k^+))\le\beta_{1k}V_1(t_k)+\beta_{2k}V_2(t_k).$$

Let $\beta_k=\max\{\beta_{1k}, \beta_{2k}\}$. Then

$$V(x(t_k^+))\le\beta_k V(t_k)\,. \tag{26}$$

When the switched system has switched k times and $t\in(t_k, t_{k+1}]$, we have

$$V(\boldsymbol{x}(t))\le\beta_1\beta_2\ldots\beta_k\mathrm{e}^{-\lambda^-T^-}\cdot\mathrm{e}^{\lambda^+T^+}V(\boldsymbol{x}(t_0)).$$

Let $\beta=\max\{\beta_1, \beta_2, \ldots, \beta_k\}$. Then

$$\begin{aligned}V(\boldsymbol{x}(t))&\le\beta^k\mathrm{e}^{-\lambda^-T^-+\lambda^+T^+}V(\boldsymbol{x}(t_0))\\&=\mathrm{e}^{-\lambda^-T^-+\lambda^+T^++k\ln\beta}V(\boldsymbol{x}(t_0))\\&\le\mathrm{e}^{-\lambda^-T^-+\lambda^+T^++k\bar{\delta}}V(\boldsymbol{x}(t_0)).\end{aligned} \tag{27}$$

From switching law S1, and noticing that $\max\{p_{k+1}-p_k\}\le\delta<+\infty$, $t\in[p_k, p_{k+1}]$, $p_0=t_0$, thus the time interval $[t_0, t]=[p_0, p_k]\cup[p_k, t]$. Then we obtain

$$\begin{aligned}&-\lambda^-T^-(t_0,t)+\lambda^+T^+(t_0,t)\\&=-\omega(t-t_0)-(\lambda^--\omega)T^-(t_0,t)\\&\quad+(\lambda^++\omega)T^+(t_0,t)\\&\le-\omega(t-t_0)-(\lambda^--\omega)T^-(p_0,p_k)\\&\quad+(\lambda^++\omega)T^+(p_0,p_k)+(\lambda^++\omega)T^+(p_k,t)\\&\le-\omega(t-t_0)+(\lambda^++\omega)T^+(p_k,t)\\&\le-\omega(t-t_0)+(\lambda^++\omega)\delta\\&=-\omega(t-t_0)+c.\end{aligned} \tag{28}$$

From inequality (28), and noticing that the selection of average dwell time satisfies Eq. (19), we obtain $V(t)\le\mathrm{e}^c\mathrm{e}^{-(\omega-\omega^*)(t-t_0)}V_{\sigma(t_0)}(t_0)$. From the conditions in Lemmas 3 and 4, for any $i\in M$, we can obtain

$$\lambda_{\min}(\boldsymbol{P}_i)\|\boldsymbol{x}_1\|^2+l_ia_{1i}\|\boldsymbol{x}_2\|^2\le V_i(\boldsymbol{x}),$$
$$V_i(\boldsymbol{x})\le\lambda_{\max}(\boldsymbol{P}_i)\|\boldsymbol{x}_1\|^2+l_ia_{2i}\|\boldsymbol{x}_2\|^2.$$

According to the definitions of k_1 and k_2,

$$V(t)\le\mathrm{e}^c\mathrm{e}^{-(\omega-\omega^*)(t-t0)}V_{\sigma(t_0)}(t_0)\Rightarrow$$
$$k_1\|\boldsymbol{x}(t)\|^2\le k_2\mathrm{e}^{c/2}\mathrm{e}^{-(\omega-\omega^*)(t-t_0)/2}\|\boldsymbol{x}(t_0)\|^2,$$

and thus $\|\boldsymbol{x}(t)\|\le\sqrt{k_2\mathrm{e}^c/k_1}\,\|\boldsymbol{x}(t_0)\|\mathrm{e}^{-\eta(t-t_0)}$; that is, the switched system is exponentially stable. This completes the proof.

3.2 Robust exponential stability for nonlinear uncertain impulsive switched systems with stable and unstable subsystems

For the subsystem of the uncertain switched system (2), we have the following results:

Lemma 5 For every $i\in M_s$, the stable nonlinear subsystem of the uncertain switched system (2)

$$\begin{cases}\dot{\boldsymbol{x}}_1(t)=(\boldsymbol{A}_{1i}+\Delta\boldsymbol{A}_{1i})\boldsymbol{x}(t)+(\boldsymbol{A}_{2i}+\Delta\boldsymbol{A}_{2i})\boldsymbol{x}_2(t),\\ \dot{\boldsymbol{x}}_2(t)=(\boldsymbol{f}_{2i}+\Delta\boldsymbol{f}_{2i})(\boldsymbol{x}_2(t)),\end{cases} \tag{29}$$

satisfies that if there are positive numbers λ^-, ε_i, η_i, and δ_i such that

(i) There exists a positive definite solution matrix $\boldsymbol{P}_i$ leading to

$$\begin{pmatrix}\boldsymbol{\Pi}_i & \boldsymbol{P}_i\boldsymbol{D}_i & \boldsymbol{P}_i\\ \boldsymbol{D}_i^{\mathrm{T}}\boldsymbol{P}_i & -(\eta_i+\delta_i)^{-1}\boldsymbol{I} & \boldsymbol{0}\\ \boldsymbol{P}_i & \boldsymbol{0} & -\varepsilon_i\boldsymbol{I}\end{pmatrix}<\boldsymbol{0}, \tag{30a}$$

where $\boldsymbol{\Pi}_i=\boldsymbol{A}_{1i}^{\mathrm{T}}\boldsymbol{P}_i+\boldsymbol{P}_i\boldsymbol{A}_{1i}+\lambda^-\boldsymbol{P}_i+\delta_i^{-1}\boldsymbol{H}_{1i}^{\mathrm{T}}\boldsymbol{H}_{1i}$.

(ii) There exists a proper, positive definite, and radically unbounded function $W_i(\boldsymbol{x}_2)$, such that

$$-d_{1i}\|\boldsymbol{x}_2\|^2\le\frac{\partial W_i(\boldsymbol{x}_2)}{\partial\boldsymbol{x}_2}\cdot\boldsymbol{f}_{2i}(\boldsymbol{x}_2)\le-b_{1i}\|\boldsymbol{x}_2\|^2, \tag{30b}$$

$$a_{1i}\|\boldsymbol{x}_2\|^2\le W_i(\boldsymbol{x}_2)\le a_{2i}\|\boldsymbol{x}_2\|^2, \tag{30c}$$

where a_{1i}, a_{2i}, b_{1i}, and d_{1i} are all positive numbers. Then the states of uncertain system (29) from time t_0 satisfy

$$V_i(t)\le\mathrm{e}^{-\lambda^-(t-t_0)}V_i(t_0),$$

where $V_i(t)=\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{P}_i\boldsymbol{x}_1+l_iW_i(\boldsymbol{x}_2)$ is a Lyapunov function candidate of system (29) with l_i being a constant to be specified in the proof.

Proof For the stable subsystem (29) with uncertainties, we choose the Lyapunov function candidate $V_i(t)=\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{P}_i\boldsymbol{x}_1+l_iW_i(\boldsymbol{x}_2)$. Then the time derivative of $V_i(t)$ along the trajectory of subsystem (29) is

$$\begin{aligned}\dot{V}_i(\boldsymbol{x}(t)) &= \dot{\boldsymbol{x}}_1^{\mathrm{T}}\boldsymbol{P}_i\boldsymbol{x}_1 + \boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{P}_i\dot{\boldsymbol{x}}_1 + l_i\frac{\partial W_i}{\partial \boldsymbol{x}_2}(\boldsymbol{x}_2)\\ &\quad\cdot(\boldsymbol{f}_{2i}(\boldsymbol{x}_2) + \Delta\boldsymbol{f}_{2i}(\boldsymbol{x}_2))\\ &= \boldsymbol{x}_1^{\mathrm{T}}(\boldsymbol{A}_{1i}^{\mathrm{T}}\boldsymbol{P}_i + \boldsymbol{P}_i\boldsymbol{A}_{1i})\boldsymbol{x}_1 + 2\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{P}_i\boldsymbol{A}_{2i}\boldsymbol{x}_2\\ &\quad + 2\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{P}_i\Delta\boldsymbol{A}_{1i}\boldsymbol{x}_1 + 2\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{P}_i\Delta\boldsymbol{A}_{2i}\boldsymbol{x}_2\\ &\quad + l_i\frac{\partial W_i}{\partial \boldsymbol{x}_2}(\boldsymbol{x}_2)\cdot(\boldsymbol{f}_{2i}(\boldsymbol{x}_2) + \Delta\boldsymbol{f}_{2i}(\boldsymbol{x}_2))\\ &= \boldsymbol{x}_1^{\mathrm{T}}(\boldsymbol{A}_{1i}^{\mathrm{T}}\boldsymbol{P}_i + \boldsymbol{P}_i\boldsymbol{A}_{1i})\boldsymbol{x}_1 + 2\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{P}_i\boldsymbol{A}_{2i}\boldsymbol{x}_2\\ &\quad + 2\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{P}_i\boldsymbol{D}_i\boldsymbol{F}_i\boldsymbol{H}_{1i}\boldsymbol{x}_1 + 2\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{P}_i\boldsymbol{D}_i\boldsymbol{F}_i\boldsymbol{H}_{2i}\boldsymbol{x}_2\\ &\quad + l_i\frac{\partial W_i}{\partial \boldsymbol{x}_2}(\boldsymbol{x}_2)\cdot(\boldsymbol{f}_{2i}(\boldsymbol{x}_2) + \Delta\boldsymbol{f}_{2i}(\boldsymbol{x}_2)).\end{aligned} \tag{31}$$

From Lemma 1, there exists positive numbers ε_i, η_i, and δ_i such that

$$2\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{P}_i\boldsymbol{A}_{2i}\boldsymbol{x}_2 \le \varepsilon_i\boldsymbol{x}_2^{\mathrm{T}}\boldsymbol{A}_{2i}^{\mathrm{T}}\boldsymbol{A}_{2i}\boldsymbol{x}_2 + \varepsilon_i^{-1}\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{P}_i\boldsymbol{P}_i\boldsymbol{x}_1, \tag{32a}$$

$$2\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{P}_i\boldsymbol{D}_i\boldsymbol{F}_i\boldsymbol{H}_{1i}\boldsymbol{x}_1 \le \delta_i^{-1}\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{H}_{1i}^{\mathrm{T}}\boldsymbol{H}_{1i}\boldsymbol{x}_1 + \delta_i\,\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{P}_i\boldsymbol{D}_i\boldsymbol{D}_i^{\mathrm{T}}\boldsymbol{P}_i\boldsymbol{x}_1, \tag{32b}$$

$$2\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{P}_i\boldsymbol{D}_i\boldsymbol{F}_i\boldsymbol{H}_{2i}\boldsymbol{x}_2 \le \eta_i^{-1}\boldsymbol{x}_2^{\mathrm{T}}\boldsymbol{H}_{2i}^{\mathrm{T}}\boldsymbol{H}_{2i}\boldsymbol{x}_2 + \eta_i\,\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{P}_i\boldsymbol{D}_i\boldsymbol{D}_i^{\mathrm{T}}\boldsymbol{P}_i\boldsymbol{x}_1. \tag{32c}$$

By the Schur complementary lemma, that the condition (i) in Lemma 5 holds is equivalent to

$$\begin{aligned}&\boldsymbol{A}_{1i}^{\mathrm{T}}\boldsymbol{P}_i + \boldsymbol{P}_i\boldsymbol{A}_{1i} + \lambda^-\boldsymbol{P}_i + \delta_i^{-1}\boldsymbol{H}_{1i}^{\mathrm{T}}\boldsymbol{H}_{1i}\\ &+(\eta_i+\delta_i)\boldsymbol{P}_i\boldsymbol{D}_i\boldsymbol{D}_i^{\mathrm{T}}\boldsymbol{P}_i + \varepsilon_i^{-1}\boldsymbol{P}_i\boldsymbol{P}_i < \boldsymbol{0}.\end{aligned} \tag{33}$$

By Eqs. (32) and (33), a simple calculation leads to

$$\begin{aligned}\dot{V}_i(\boldsymbol{x}(t)) &\le -\lambda^-\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{P}_i\boldsymbol{x}_1 + \varepsilon_i\boldsymbol{x}_2^{\mathrm{T}}\boldsymbol{A}_{2i}^{\mathrm{T}}\boldsymbol{A}_{2i}\boldsymbol{x}_2\\ &\quad + \eta_i^{-1}\boldsymbol{x}_2^{\mathrm{T}}\boldsymbol{H}_{2i}^{\mathrm{T}}\boldsymbol{H}_{2i}\boldsymbol{x}_2 - l_ib_{2i}\|\boldsymbol{x}_2\|^2 + l_iq_id_{1i}\|\boldsymbol{x}_2\|^2\\ &\le -\lambda^-\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{P}_i\boldsymbol{x}_1 - \lambda^-l_iW_i(\boldsymbol{x}_2) + \lambda^-l_iW_i(\boldsymbol{x}_2)\\ &\quad + \varepsilon_i\lambda_{\max}(\boldsymbol{A}_{2i}^{\mathrm{T}}\boldsymbol{A}_{2i})\|\boldsymbol{x}_2\|^2 + \eta_i^{-1}\lambda_{\max}(\boldsymbol{H}_{2i}^{\mathrm{T}}\boldsymbol{H}_{2i})\|\boldsymbol{x}_2\|^2\\ &\quad - l_ib_{2i}\|\boldsymbol{x}_2\|^2 + l_iq_id_{1i}\|\boldsymbol{x}_2\|^2\\ &\le -\lambda^-V_i(\boldsymbol{x}) - [l_i(b_{2i} - \lambda^-l_ia_{2i} - q_id_{1i})\\ &\quad - \varepsilon_i\lambda_{\max}(\boldsymbol{A}_{2i}^{\mathrm{T}}\boldsymbol{A}_{2i}) - \eta_i^{-1}\lambda_{\max}(\boldsymbol{H}_{2i}^{\mathrm{T}}\boldsymbol{H}_{2i})]\|\boldsymbol{x}_2\|^2.\end{aligned} \tag{34}$$

Let

$$\begin{aligned}l_i &\ge [\varepsilon_i\lambda_{\max}(\boldsymbol{A}_{2i}^{\mathrm{T}}\boldsymbol{A}_{2i}) + \eta_i^{-1}\lambda_{\max}(\boldsymbol{H}_{2i}^{\mathrm{T}}\boldsymbol{H}_{2i})]\\ &\quad\cdot(b_{2i} - \lambda^-l_ia_{2i} - q_id_{1i})^{-1}.\end{aligned}$$

Then

$$\dot{V}_i(\boldsymbol{x}(t)) \le -\lambda^-V_i(\boldsymbol{x}(t)) \Rightarrow V_i(\boldsymbol{x}(t)) \le V_i(\boldsymbol{x}(t_0))\mathrm{e}^{-\lambda^-(t-t_0)}. \tag{35}$$

Lemma 6 For every $i\in M_{\mathrm{u}}$, the unstable uncertain nonlinear subsystem

$$\begin{cases}\dot{\boldsymbol{x}}_1(t) = (\boldsymbol{A}_{1i} + \Delta\boldsymbol{A}_{1i})\boldsymbol{x}(t) + (\boldsymbol{A}_{2i} + \Delta\boldsymbol{A}_{2i})\boldsymbol{x}_2(t),\\ \dot{\boldsymbol{x}}_2(t) = (\boldsymbol{f}_{2i} + \Delta\boldsymbol{f}_{2i})(\boldsymbol{x}_2(t)),\end{cases} \tag{36}$$

satisfies that if there exists positive numbers λ^+, ε_i, η_i, and δ_i such that

(ia) There exists a positive definite solution matrix $\boldsymbol{P}_i$ leading to

$$\begin{pmatrix}\overline{\boldsymbol{\Pi}}_i & \boldsymbol{P}_i\boldsymbol{D}_i & \boldsymbol{P}_i\\ \boldsymbol{D}_i^{\mathrm{T}}\boldsymbol{P}_i & -(\eta_i+\delta_i)^{-1}\boldsymbol{I} & \boldsymbol{0}\\ \boldsymbol{P}_i & \boldsymbol{0} & -\varepsilon_i\boldsymbol{I}\end{pmatrix} < \boldsymbol{0}, \tag{37a}$$

where $\overline{\boldsymbol{\Pi}}_i = \boldsymbol{A}_{1i}^{\mathrm{T}}\boldsymbol{P}_i + \boldsymbol{P}_i\boldsymbol{A}_{1i} - \lambda^+\boldsymbol{P}_i + \delta_i^{-1}\boldsymbol{H}_{1i}^{\mathrm{T}}\boldsymbol{H}_{1i}$.

(iia) There exists a proper, positive definite, and radically unbounded function $W_i(\boldsymbol{x}_2)$, such that

$$\frac{\partial W_i(\boldsymbol{x}_2)}{\partial \boldsymbol{x}_2}\cdot\boldsymbol{f}_{2i}(\boldsymbol{x}_2) \le b_{2i}\|\boldsymbol{x}_2\|^2, \tag{37b}$$

$$a_{1i}\|\boldsymbol{x}_2\|^2 \le W_i(\boldsymbol{x}_2) \le a_{2i}\|\boldsymbol{x}_2\|^2, \tag{37c}$$

where a_{1i}, a_{2i}, and b_{2i} are all positive numbers. Then the states of system (36) from time t_0 satisfy

$$V_i(t)\le \mathrm{e}^{\lambda^+(t-t_0)}V_i(t_0),$$

where $V_i(t)=\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{P}_i\boldsymbol{x}_1+l_iW_i(\boldsymbol{x}_2)$ is a Lyapunov function candidate of system (36) with l_i being a constant to be specified in the proof.

Proof For unstable uncertain subsystem (36), we choose the following Lyapunov function candidate:

$$V_i(t)=\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{P}_i\boldsymbol{x}_1+l_iW_i(\boldsymbol{x}_2).$$

Similar to Lemma 4, the time derivative of $V_i(t)$ along the trajectory of subsystem (36) is

$$\begin{aligned}\dot{V}_i(\boldsymbol{x}(t)) &= \boldsymbol{x}_1^{\mathrm{T}}(\boldsymbol{A}_{1i}^{\mathrm{T}}\boldsymbol{P}_i + \boldsymbol{P}_i\boldsymbol{A}_{1i})\boldsymbol{x}_1 + 2\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{P}_i\boldsymbol{A}_{2i}\boldsymbol{x}_2\\ &\quad + 2\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{P}_i\boldsymbol{D}_i\boldsymbol{F}_i\boldsymbol{H}_{1i}\boldsymbol{x}_1 + 2\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{P}_i\boldsymbol{D}_i\boldsymbol{F}_i\boldsymbol{H}_{2i}\boldsymbol{x}_2\\ &\quad + l_i\frac{\partial W_i}{\partial \boldsymbol{x}_2}(\boldsymbol{x}_2)\cdot(\boldsymbol{f}_{2i}(\boldsymbol{x}_2) + \Delta\boldsymbol{f}_{2i}(\boldsymbol{x}_2)).\end{aligned} \tag{38}$$

By the Schur complementary lemma, that condition (ia) in Lemma 6 holds is equivalent to

$$\boldsymbol{A}_{1i}^{\mathrm{T}}\boldsymbol{P}_i+\boldsymbol{P}_i\boldsymbol{A}_{1i}-\lambda^+\boldsymbol{P}_i+\delta_i^{-1}\boldsymbol{H}_{1i}^{\mathrm{T}}\boldsymbol{H}_{1i}$$
$$+(\eta_i+\delta_i)\boldsymbol{P}_i\boldsymbol{D}_i\boldsymbol{D}_i^{\mathrm{T}}\boldsymbol{P}_i+\varepsilon_i^{-1}\boldsymbol{P}_i\boldsymbol{P}_i<\boldsymbol{0}. \quad (39a)$$

According to Eqs. (32), (37), and (39a), Eq. (38) can be changed to

$$\dot{V}_i(\boldsymbol{x}(t))\le\lambda^+\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{P}_i\boldsymbol{x}_1+\lambda^+l_iW_i(\boldsymbol{x}_2)-\lambda^+l_iW_i(\boldsymbol{x}_2)$$
$$+[\varepsilon_i\lambda_{\max}(\boldsymbol{A}_{2i}^{\mathrm{T}}\boldsymbol{A}_{2i})+\eta_i^{-1}\lambda_{\max}(\boldsymbol{H}_{2i}^{\mathrm{T}}\boldsymbol{H}_{2i})$$
$$+l_ib_{2i}+l_ib_{2i}q_i]\|\boldsymbol{x}_2\|^2, \quad (39b)$$

$$\dot{V}_i(\boldsymbol{x}(t))\le\lambda^+V_i(\boldsymbol{x}(t))-[l_i(\lambda^+a_{1i}-b_{2i}-b_{2i}q_i)$$
$$-\varepsilon_i\lambda_{\max}(\boldsymbol{A}_{2i}^{\mathrm{T}}\boldsymbol{A}_{2i})-\eta_i^{-1}\lambda_{\max}(\boldsymbol{H}_{2i}^{\mathrm{T}}\boldsymbol{H}_{2i})]\|x_2\|^2. \quad (39c)$$

Let

$$l_i\ge[\varepsilon_i\lambda_{\max}(\boldsymbol{A}_{2i}^{\mathrm{T}}\boldsymbol{A}_{2i})+\eta_i^{-1}\lambda_{\max}(\boldsymbol{H}_{2i}^{\mathrm{T}}\boldsymbol{H}_{2i})]$$
$$\cdot(\lambda^+a_{1i}-b_{2i}-b_{2i}q_i)^{-1}.$$

Then

$$\dot{V}_i(\boldsymbol{x}(t))\le\lambda^+V_i(\boldsymbol{x}(t))\Rightarrow V_i(\boldsymbol{x}(t))\le V_i(\boldsymbol{x}(t_0))\mathrm{e}^{\lambda^+(t-t_0)}.$$

Thus, we can obtain a sufficient condition of robust exponential stability for the uncertain nonlinear impulsive switched system (2) with stable and unstable subsystems by Theorem 2.

Theorem 2 For the nonlinear impulsive uncertain switched system (2) with stable and unstable subsystems, the uncertainties satisfy Assumptions 1 and 2. When $i\in M_{\mathrm{s}}$, the subsystems are stable and meet conditions (i) and (ii) in Lemma 5; When $i\in M_{\mathrm{u}}$, the subsystems are unstable and meet conditions (ia) and (iia) in Lemma 6. The nonlinear impulsive increments satisfy Assumption 3, and the switching law is carried out according to S1. Then the uncertain switched system is exponentially stable. The attenuation of the states satisfies

$$\|\boldsymbol{x}(t)\|\le\sqrt{k_2\mathrm{e}^c/k_1}\,\|\boldsymbol{x}(t_0)\|\mathrm{e}^{-\eta(t-t_0)}. \quad (40)$$

All the symbols have the same meanings as those in Theorem 1.

Proof We choose the piecewise Lyapunov function candidate for uncertain switched system (2) as

$$V_{\sigma(t)}(t)=\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{P}_{\sigma(t)}\boldsymbol{x}_1+l_{\sigma(t)}W_{\sigma(t)}(\boldsymbol{x}_2). \quad (41)$$

When $t\in(t_k, t_{k+1}]$, the uncertain switched system switches to the i_kth subsystem; that is, the i_kth subsystem is active during this time interval. Without loss of generality, we assume that the i_kth subsystem is stable. Then from Lemma 5, we obtain

$$V_{i_k}(\boldsymbol{x}(t))\le V_{i_k}(\boldsymbol{x}(t_k))\mathrm{e}^{-\lambda^-(t-t_k)}.$$

When $t\in(t_{k+1}, t_{k+2}]$, the uncertain switched system switches to the i_{k+1}th subsystem. Assume that the i_{k+1}th subsystem is unstable. Then from Lemma 6, we have

$$V_{i_{k+1}}(\boldsymbol{x}(t))\le V_{i_{k+1}}(\boldsymbol{x}(t_{k+1}))\mathrm{e}^{\lambda^+(t-t_{k+1})}.$$

Then we can use the similar method in Theorem 1 to prove Theorem 2.

Remark 2 The parameters (such as $\boldsymbol{P}_i$) in Theorem 2 are different from those in Theorem 1, because they are satisfying different inequalities. We use the same notations here just for simplification.

4 Simulation

Consider the following nonlinear impulsive switched system consisting of two subsystems:

$$\begin{cases}\dot{\boldsymbol{x}}_1(t)=\boldsymbol{A}_{1\sigma(t)}\boldsymbol{x}(t)+\boldsymbol{A}_{2\sigma(t)}\boldsymbol{x}_2(t),\\ \dot{\boldsymbol{x}}_2(t)=\boldsymbol{f}_{2\sigma(t)}(\boldsymbol{x}_2(t)),\end{cases} \quad (42)$$

where $\sigma(t)$: $[0, +\infty)\rightarrow M=\{1, 2\}$,

$$\boldsymbol{A}_{11}=\begin{bmatrix}-2&1\\0&-2\end{bmatrix},\ \boldsymbol{A}_{21}=\begin{bmatrix}0\\1\end{bmatrix},$$
$$\boldsymbol{A}_{12}=\begin{bmatrix}1&0\\1&1\end{bmatrix},\ \boldsymbol{A}_{22}=\begin{bmatrix}-1\\0\end{bmatrix},$$
$$\boldsymbol{f}_{21}=-2\boldsymbol{x}_2-\boldsymbol{x}_2\sin^2\boldsymbol{x}_2,\ \boldsymbol{f}_{22}=\boldsymbol{x}_2-\boldsymbol{x}_2\sin^2\boldsymbol{x}_2.$$

The impulsive increments at time $t=t_k$ are

$$\boldsymbol{E}_{1k}=\begin{pmatrix}-0.5&0\\0&0.5\end{pmatrix},\ \boldsymbol{\phi}_k=\begin{pmatrix}0.1\sin(x_{11}(t_k))\\0.2\sin(x_{12}(t_k))\end{pmatrix},$$
$$\boldsymbol{E}_{2k}=-0.3,\ \boldsymbol{\phi}_{2k}=0.1\sin^2(x_2(t_k)).$$

We can select $W_1(x_2)=W_2(x_2)=0.5x_2^2$. Then by direct calculation we can obtain

$$\frac{\partial W_1(\boldsymbol{x}_2)}{\partial\boldsymbol{x}_2}\cdot\boldsymbol{f}_{21}(\boldsymbol{x}_2)=-2\boldsymbol{x}_2^2-\boldsymbol{x}_2^2\sin^2\boldsymbol{x}_2\le-2\|\boldsymbol{x}_2\|^2,$$

$$\frac{\partial W_2(\boldsymbol{x}_2)}{\partial \boldsymbol{x}_2}\cdot \boldsymbol{f}_{22}(\boldsymbol{x}_2)=\boldsymbol{x}_2^2-\boldsymbol{x}_2^2\sin^2\boldsymbol{x}_2\le\|\boldsymbol{x}_2\|^2.$$

Thus, the given subsystems satisfy the conditions in Theorem 1. Let λ^{-}=3.6 and λ^{+}=2.5. By solving LMI (9a), we can obtain the solution matrices and parameters:

$$\boldsymbol{P}_1=\begin{pmatrix}0.4949 & -0.9763\\ -0.9763 & 6.2636\end{pmatrix},$$

$$\boldsymbol{P}_2=\begin{pmatrix}2.1501 & -0.3974\\ -0.3974 & 0.2618\end{pmatrix},$$

$$\varepsilon_1=16.6124,\ \varepsilon_2=4.8372.$$

It is easy to know that the first subsystem is stable, and the second subsystem is unstable. We choose the switching period as 0.8 s (which satisfies the average dwell time $T_a>T_a^*=0.383$). For every period, let the first subsystem be active for 0.6 s, and the second subsystem be active for 0.2 s. Take the initial state condition as $\boldsymbol{x}_0=[1\ -2\ 2]^{\mathrm{T}}$. Figs. 1 and 2 show the state curves of the switched system. The impulsive increments at the switching time instant have strong influences on the state curves. The impulsive increments not only prolong the convergent time, but also increase the wave character of the system. In fact, if we choose the impulsive increments on overlarge, the system will be unstable.

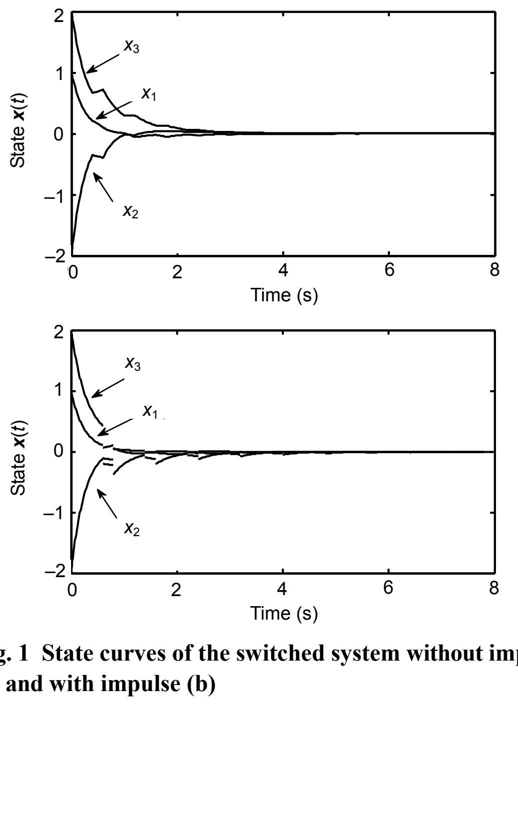

Fig. 1 State curves of the switched system without impulse (a) and with impulse (b)

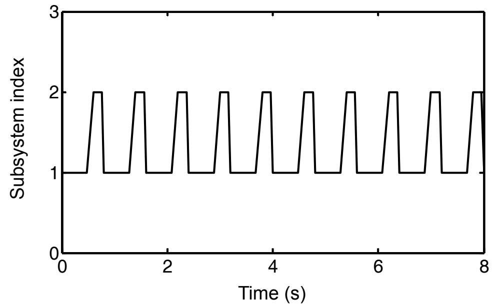

Fig. 2 Switching the two subsystems

5 Conclusions

In this paper we investigate the stability and robust stability of nonlinear impulsive switched systems consisting of both stable and unstable subsystems. Every subsystem of the switched system is a nonlinear system. The impulsive increments considered in this study are chosen as nonlinear functions of the systems' states. We have extended the impulsive increments for linear subsystems to nonlinear subsystems of switched systems.

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